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MECHANICAL RELATIONSHIP OF REINFORCEMENTS AND THE BINDER MATRIX

Final Report

By

Juan Haener

September 1965

U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA

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WHITTAKER CORPORATION

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Work on this project was carried out by the Whittaker Corporation, Narmco Research and Development Division, under the overall direction of Dr. Juan Haener.

The purpose of this study is to gain a better understanding of the mechanical behavior of fiber reinforced plastic composites and of the interaction between the fiber reinforcement and the binder material. The experimental effort which is related to this theoretical work is being accomplished under Contract DA 44-177-AMC-320(T).

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MECHANICAL RELATIONSHIP OF REINFORCEMENTS AND THE BINDER MATRIX

Final Report

by

Juan Haener

Prepared by
WHITTAKER CORPORATION
Narmco Research & Development Division
San Diego, California

for

U.S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA

ABSTRACT

The mechanical interactions between reinforcement and matrix in a composite under load have been described by stress distribution equations in the fibers and in the matrix. Although special attention has been given to stresses resulting from polymerization and temperature shrinkage, the solutions obtained can be easily adapted for use with the boundary conditions of external loads. Several cases were considered:

(1) a cylindrical filament of finite length embedded in a resin cylinder;

(2) a cylindrical filament of infinite length embedded in a resin cylinder; and

(3) a matrix supporting a central fiber surrounded by six symmetrically spaced fibers, each in turn surrounded by six others.

FOREWORD

This final report was prepared by Whittaker Corporation, Narmco Research & Development Division, San Diego, California, under Contract DA 44-177-AMC-208(T) entitled "Mechanical Relationship of Reinforcements and the Binder Matrix." The work is being accomplished under the direction of R. P. McKinnon, Contracting Officer, U.S. Army Aviation Materiel Laboratories (USAAVLABS), Fort Eustis, Virginia.

This report covers work conducted from 10 June 1964 through 1 March 1965.

Work on this project was conducted under the overall direction of Dr. Juan Haener. Dr. Gerhard Nowak contributed as a consultant to this program. Report has been reviewed and approved by B. Levenetz, Research Engineering; R. Hidde, Project Office; and B. Duft, Manager, Engineering Department.

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SYMBOLS

a_o, a	radius of fiber before and after stress develops respectively
$2a_o$	radius of reinforcements before stress develops
a, A, A_i	constants
b	diameter of resin
$2b_o$	radius of resin cylinder before stress develops
B_i	constants
B_{ik}	constants defined in equations (182) through (217)
C_i	constants
D_i	constants defined in equations (218) through (223) and equations (269) through (274)
D_{ij}, D^{ij}	local strain tensor including polymerization or temperature contraction
D_{ii}	volumetric change, or dilatation, or trace of the strain tensor
D'_{ij}	strain tensor, excluding polymerization or temperature contractions
e^i	unit vector in the direction of particle length
e^P	unit vector originating at point P
e^q	unit vector originating at point Q
E	Young's modulus of elasticity
f_i	defined in equations (224) through (268)
df^i	area element
g^i	body force density
dG^i	body force density
g_{ji}	gage tensor
H	Hankel functions
k_1, k_2	defined in equations (178) and (179)

SYMBOLS (Continued)

$\mathcal{l}_o$	length of specimen before stress develops
$\mathcal{l}_I$, $\mathcal{l}^I$	length after shrinkage
m_1, m_2	defined in equations (180) and (181)
n_i, n^i	vector normal to the particle area in reinforcement or matrix
p_i, p^i	stress vector perpendicular to area element df_i
q	index referring to point Q
r	cylindrical coordinate in the radial direction
R	function of the corrdinate r
s	original length of the particle observed
$\bar{s}$	length of the particle after deformation
ds_j	component of dx_i (u_j) along t_{ji}
t_{ji}	unit vector along the u_j coordinate (tangential to u_j)
T	temperature
u_j, u_1, u_2, u_3	general coordinates of original position of points
$\bar{u}_j$, $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$	general coordinates of displaced points
x_i	location vector of point in an unstrained medium
x_i (u_j)	location vector, expressed in curvilinear coordinates of point in an unstrained medium
X^j, X_j	principal stress, or eigen directions
z	cylindrical coordinates in axial direction (see Figure 6)
α^I, α^{II}	coefficients of thermal expansion
η	coefficient of shrinkage due to reasons other than thermal (e.g., polymerization)
β^I, β^{II}, or β_1, β_2	coefficient of shrinkage
β, β_p	eigenvalue defined in equation (430)

SYMBOLS (Continued)

δ_i	defined in equations (225) through (233)
δ_{ij}, δ_p^q	Knonecker delta
Δ_{ij}	strain tensor due to expansion or contraction, including shrinkage due to polymerization
λ	a Lamé constant
λ	eigenvalue of the wave (defined in the Appendix)
$\Lambda_{ijk\ell}$	tensor of elasticity
μ	a Lamé constant
ν	Poisson ratio
ξ^i, ξ_i	displacements in direction physical components of the displacement vector
σ^{ij}, σ_{ij}	stress tensor component
σ_{11}	stress in radial direction
σ_{12}	shear in tangential direction
σ_{13}	shear in axial direction
σ_{21}	shear in radial direction
σ_{22}	stress in tangential direction
σ_{23}	shear in axial direction
σ_{31}	shear in radial direction
σ_{32}	shear in tangential direction
σ_{33}	stress in axial direction
φ ϕ	cylindrical coordinate
Φ	potential function

SYMBOLS (Continued)

Superscripts:

I	refers to reinforcement
II	refers to matrix

Subscripts:

i, j, k, ℓ, r, or s	free index
$\vec{I}_i$	a set of orthogonal unit vectors
g_j	metric vectors tangents to the curvilinear coordinate elements
du_j	curvilinear coordinate elements Figure (1.b)
$G_{\ell m}$	metric tensor of the strained body
ds	differential arc length
$\vec{V}$	displacement vector

SUMMARY

This program was initiated to provide further understanding of the mechanical interaction between reinforcements and the matrix in a fibrous, composite material. The mathematical fundamentals initially established were a set of solutions of partial differential equation describing the distortions in both the reinforcement and matrix. The solutions were kept in general form so that they could be used in establishing those internal stress distributions resulting from polymerization and temperature shrinkages as well as those resulting from loads imposed at the outer surface.

Two cases are considered: (1) a cylindrical filament of finite length centrally embedded in a matrix cylinder, and (2) a matrix which supports a central fiber that is surrounded by six symmetrically spaced fibers.

Undulating stresses apparently result in both cases. Complete axial symmetry was assumed in the monofilament case, while for the seven-fiber model, the stress pattern is repeating six times around the central fiber.

The belief is that the first case has been solved rigorously in this work without any simplifying approximations or assumptions. For the latter case, certain potential functions representing the component of the curl of the displacement vector and one function of volumetric compressibility have been set to zero; to date, however, there is no rigorous theoretical support for this assumption.

INTRODUCTION

The strength of composites appears to be well below that which might be achieved, when the available strength of the reinforcing filaments is considered.* Any attempt to develop reinforced composites with strength-to-weight-ratios higher than those presently attainable must be based on more complete information concerning the mechanical interaction between the reinforcement and matrix in a composite. The subsequent text summarizes the mathematical development accomplished under a program whose goal was to more clearly define this interaction.

The initial consideration is that of distortions in two homogeneous materials, one embedded in the other. A consideration of equilibrium, a stress-strain relation, and a strain-displacement relation will lead to differential equations for the stresses, strains, and displacements on and within a solid body. As a specific case, a cylindrical system is considered in some detail. This basic development is then applied to two cases.

Case 1 - A cylindrical filament centrally embedded in a matrix cylinder is studied. Axial symmetry is assumed. The mathematically rigorous solution of the three differential equations of displacement is derived. The proper boundary conditions applied to the solutions results in the equation for strain and stress distributions in the composite. Special attention is given to the boundary conditions on the interface between reinforcement and matrix, and to the case where external forces are applied. In this manner the residual stress distribution in the composite is obtained. A specific numerical example of an infinite fiber surrounded by resin is given.

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* Narmco Research & Development, Potential of Filament-Wound Composites, Final Report, Contract NOw 61-0623-c (FBM), San Diego, Calif., Mar 1962, p. 1.

Case 2 - A matrix supporting a central fiber that is surrounded by six symmetrically spaced fibers is treated. The mathematical steps are similar to those followed in the first case, but are much more complicated as there is only hexagonal symmetry around the fibers. This and other geometric limitations are assumed as boundary conditions. The boundary conditions on the interface between the reinforcement and the composite are the same as those in the single-fiber case. The basic equilibrium equations (expressed in terms of displacements) are considered simultaneously, and general solutions for the displacements or distortions in the composite are obtained.

TECHNICAL DISCUSSION

DEVELOPMENT OF BASIC EQUATIONS

The component materials of the composite are assumed to be under internal and external stresses. Included are polymerization shrinkages and temperature expansion or contraction phenomena. The total strain resulting from such stresses is represented by the strain tensor.*

In cylindrical coordinates, the elements of the strain tensor are obtained.

$$D_{11} = \frac{\partial \xi_1}{\partial r} \tag{1}$$

$$D_{22} = \frac{1}{r}\left[\xi_1 + \frac{\partial \xi_2}{\partial \varphi}\right] \tag{2}$$

$$D_{33} = \frac{\partial \xi_3}{\partial z} \tag{3}$$

$$D_{12} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial \xi_1}{\partial \varphi} + \frac{\partial \xi_2}{\partial r} - \frac{1}{r}\xi_2\right] \tag{4}$$

$$D_{23} = \frac{1}{2}\left[\frac{\partial \xi_2}{\partial z} + \frac{1}{r}\frac{\partial \xi_3}{\partial \varphi}\right] \tag{5}$$

$$D_{31} = \frac{1}{2}\left[\frac{\partial \xi_3}{\partial r} + \frac{\partial \xi_1}{\partial z}\right] \tag{6}$$

* Equation (A71) (A72) in the Appendix to this report.

Equations (1) through (5) introduced into the stress tensor*

$$\sigma^{ij} = \frac{E}{1+\nu}\left\{L^{ij} + \delta_{ij}\left(\frac{\nu}{1-2\nu} D^{kk} - \frac{1+\nu}{1-2\nu}\left[\alpha T + \beta\right]\right)\right\} \tag{7}$$

result in stress distribution in radial, tangential, and axial direction, as symbolized in the following σ_{11} , σ_{22} , σ_{33} direct stresses in radial, tangential, and axial direction, and σ_{12} , σ_{13} , σ_{23} , the shear stresses in tangential, axial, and radial direction.

(Note: All of the stresses are written with subscripts, which is customary for the theory of elasticity. Superscripts I and II indicate the stress in the reinforcements and resin respectively.)

To obtain the displacement ξ_1^I , ξ_2^I , ξ_3^I and ξ_1^{II} , ξ_2^{II} , ξ_3^{II} the differential equations of displacements** must be solved. Restated, the partial differential equations are

In the radial direction:

$$\frac{1-\nu}{1-2\nu}\frac{\partial^2\xi_1}{\partial r^2} + \frac{1}{2r^2}\frac{\partial^2\xi_1}{\partial\varphi^2} + \frac{1}{2}\frac{\partial^2\xi_1}{\partial z^2} + \frac{1}{2(1-2\nu)}\frac{1}{r}\frac{\partial^2\xi_2}{\partial r\,\partial\varphi} +$$

$$\frac{1}{2(1-2\nu)}\frac{\partial^2\xi_3}{\partial r\,\partial z} - \frac{3-4\nu}{2(1-2\nu)}\frac{1}{r^2}\frac{\partial\xi_2}{\partial\varphi} + \frac{1-\nu}{1-2\nu}\frac{1}{r}\left(\frac{\partial\xi_1}{\partial r} - \frac{1}{r}\xi_1\right) -$$

$$\frac{1+\nu}{1-2\nu}\left(\alpha\frac{\partial T}{\partial r} + \frac{\partial\beta}{\partial r}\right)\frac{1+\nu}{E}\left(g_1\cos\varphi - g_2\sin\varphi\right) = 0 \tag{8}$$

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* Equation (63) in the Appendix

** Equations (73) in the Appendix

In the tangential direction:

$$\frac{1}{2}\frac{\partial^2 \xi_2}{\partial r^2} + \frac{1-\nu}{1-2\nu}\frac{1}{r^2}\frac{\partial^2 \xi_2}{\partial \varphi^2} + \frac{1}{2}\frac{\partial^2 \xi_2}{\partial z^2} + \frac{1}{2(1-2\nu)}\frac{1}{r}\frac{\partial^2 \xi_1}{\partial r\,\partial \varphi} +$$

$$\frac{1}{2(1-2\nu)}\frac{1}{r}\frac{\partial^2 \xi_3}{\partial \varphi\,\partial z} + \frac{1}{2r}\left(\frac{1}{r}\frac{3-4\nu}{1-2\nu}\frac{\partial \xi_1}{\partial \varphi} + \frac{\partial \xi_2}{\partial r} - \frac{1}{r}\xi_2\right) =$$

$$\frac{1+\nu}{1-2\nu}\frac{1}{r}\left(\alpha\frac{\partial T}{\partial \varphi} + \frac{\partial \beta}{\partial \varphi}\right) + \left(g_1 \sin\varphi + g_2 \cos\varphi\right)\frac{1+\nu}{E} = 0 \qquad (9)$$

In the axial direction:

$$\frac{1}{2}\frac{\partial^2 \xi_3}{\partial r^2} + \frac{1}{2r^2}\frac{\partial^2 \xi_3}{\partial \varphi^2} + \frac{1-\nu}{1-2\nu}\frac{\partial^2 \xi_3}{\partial z^2} + \frac{1}{2(1-2\nu)}\frac{\partial^2 \xi_1}{\partial r\,\partial z} +$$

$$\frac{1}{2(1-2\nu)}\frac{1}{r}\frac{\partial^2 \xi_2}{\partial \varphi\,\partial z} + \frac{1}{2(1-2\nu)}\frac{1}{r}\frac{\partial \xi_1}{\partial z} + \frac{1}{2r}\frac{\partial \xi_3}{\partial r} -$$

$$\frac{1+\nu}{1-2\nu}\left(\alpha\frac{\partial T}{\partial z} \quad \frac{\partial \beta}{\partial z}\right) + g_3\frac{1+\nu}{E} = 0 \qquad (10)$$

It can be seen that the three differential equations are coupled partial differential equations which, by introduction of adequate mathematical potential, can be separated. Before this is considered, however, an extensive study of the boundary conditions is presented to provide insight to the problem and facilitate its solutions. The first is a finite fiber in a matrix cylinder, and the second condition consists of a matrix and a cent-al fiber surrounded by six symmetrically placed fibers.

BOUNDARY CONDITIONS OF A SINGLE FIBER OF FINITE LENGTH IN A MATRIX

Consider a fiber of diameter $2a_o$ and length $2\ell_o$ embedded in a matrix cylinder of outer diameter $2b_o$ and length $2\ell_o$. The index zero refers to dimensions before any stress has developed.

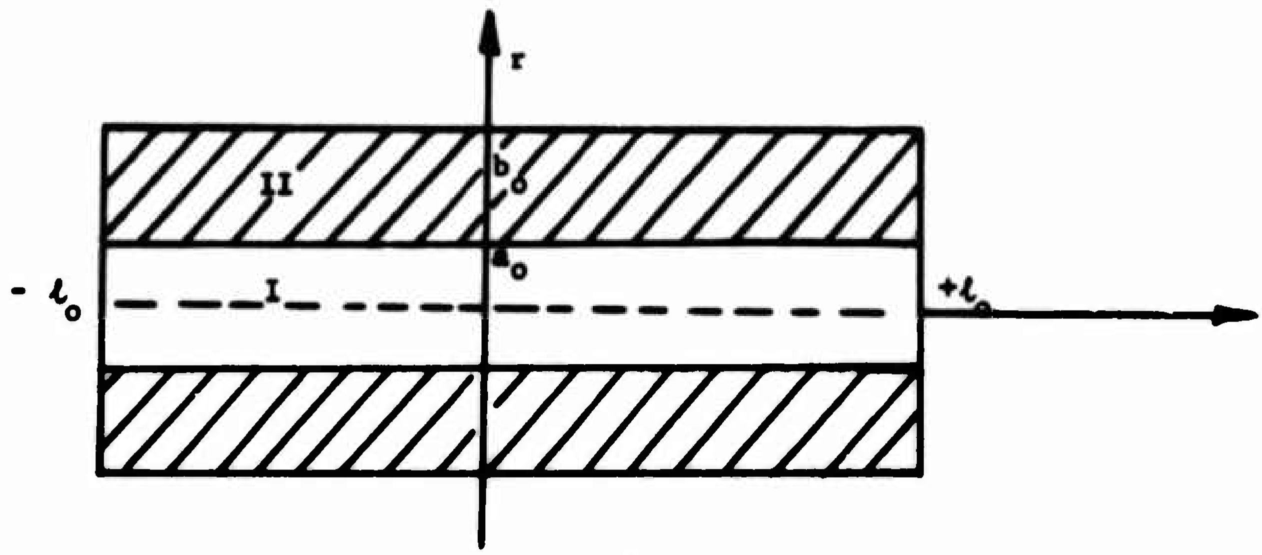


Figure 1

The system of the partial differential equations (8) through (10) describes the distortion in both materials. It is

$$\xi_i^{I}(r, z) \tag{11}$$

the distortion vector in the fiber and

$$\xi_i^{II}(r, z) \tag{12}$$

the distortion vector in the matrix. The material constants E, ν, β, α, which have superscripts I or II refer to the reinforcement or matrix respectively.

The fundamental equations of the theory of elasticity are valid and the results obtained in the Appendix can be utilized. The boundary conditions are implemented by the fact that both ends are free from stresses.

If there are no external forces, then

$$\sigma_{ij}^{II}\left(b_{II}\ ,\ z\right)n_j\ =\ 0 \tag{13}$$

is considered a set of boundary conditions. In equation (13),

$$\sigma_{ij}^{II}\left(b_{II}\ ,\ z\right)$$

is the stress tensor (equation 7) on the surface of the matrix in the points

$$\left(r\ =\ b_{II}\ =\ b_o\left(1\ -\ \beta^{II}\right)\ ;\ z\right)$$

and n_j is the unit vector normal to the surface element in the point b_I , z .

The length $b_{II}\ =\ b_o\left(1\ -\ \beta^{II}\right)$, the outer radius shortened by the shrinkage of the matrix. But since n_j is always in 1 (or r) direction, it is, in the assumed cylindrical coordinate system,

$$n_j\ =\ e_{1j}\ =\ \delta_{1j} \tag{14}$$

so that equation (13) becomes

$$\sigma_{ij}^{II}\left(b_{II}\ ,\ z^{II}\right)\delta_{1j}\ =\ \sigma_{i1}^{II}\left(b_{II}\ ,\ z^{II}\right)\ =\ 0 \tag{15}$$

or, more specifically,

$$\sigma_{11}^{II}\left(b_{II}, z\right) = 0 \qquad (16)$$

$$\sigma_{21}^{II}\left(b_{II}, z\right) = 0 \qquad (17)$$

$$\sigma_{31}\left(b_{II}, z\right) = 0 \qquad (18)$$

By using the equation for the stress tensor, these boundary conditions result in

$$D_{11}^{II}\left(b_{II}, z\right) + \frac{\nu^{II}}{1 - 2\nu^{II}} D_{KK}^{II}\left(b_{II}, z\right) = 0 \qquad (19)$$

$$D_{21}^{II}\left(b_{II}, z\right) = 0 \qquad (20)$$

$$D_{31}^{II}\left(b_{II}, z\right) = 0 \qquad (21)$$

Using equations (1) through (6), equations (19) through (21) become

$$\frac{\partial \xi_1^{II}\left(b_{II}, z\right)}{\partial r} + \frac{\nu^{II}}{1 - 2\nu^{II}}\left[\frac{1}{b_{II}} \frac{\partial\left(r\xi_1^{II}\right)}{\partial_{II}}\left(b_{II}, z\right) + \frac{\partial \xi_3^{II}\left(b_{II}, z\right)}{\partial z}\right] = 0 \qquad (22)$$

$$\frac{\partial}{\partial r}\left(\frac{\xi_2^{II}}{r}\right)\left(b_{II}\ ,\ z\right) = 0 \qquad (23)$$

$$\frac{\partial \xi_3^{II}}{\partial r}\left(b_{II}\right),\ z + \frac{\partial \xi_1^{II}}{\partial z}\left(b_{II}\ ,\ z\right) = 0 \qquad (24)$$

Equations (22) through (24) can also be used for the infinite fiber.

The boundary conditions for the fiber at the end of the specimen are

$$\left.\begin{aligned} \sigma_{ij}^{I}\left(r\ ,\ \ell_I\right) n_j &= 0 \\ \sigma_{ij}^{I}\left(r\ ,-\ell_I\right) n_j &= 0 \end{aligned}\right\} \text{ for } 0 \le r \le a \qquad (25)$$

and, for the matrix, are

$$\left.\begin{aligned} \sigma_{ij}^{II}\left(r\ ,\ \ell_{II}\right) n_j &= 0 \\ \sigma_{ij}^{II}\left(r\ ,\ \ell_{II}\ n_j\right) &= 0 \end{aligned}\right\} \text{ for } a \le r \le b \qquad (26)$$

In equations (25) and (26), consideration must be given to the fact that the length ℓ in both cases is subjected to shrinkage, so that

$$\ell_I = \ell_o\left(1 - \beta^I\right)$$

and

$$\ell_{II} = \ell_o\left(1 - \beta^{II}\right)$$

Here, the normal vector is in z direction (axial)

$$n_j = e_{3j} = \delta_{3j}$$

Equations (25) and (26) result in the conditions for the reinforcement:

$$\sigma_{33}^{I}\left(r\ ,\ \ell_I\right) = -\,\sigma_{33}^{I}\left(r\ ,\ -\ \ell_I\right) = 0 \tag{27}$$

$$\sigma_{32}^{I}\left(r\ ,\ \ell_I\right) = -\,\sigma_{32}^{I}\left(r\ ,\ -\ \ell_I\right) = 0 \tag{28}$$

$$\sigma_{31}^{I}\left(r\ ,\ \ell_I\right) = +\,\sigma_{31}^{I}\left(r\ ,\ -\ \ell_I\right) = 0 \tag{29}$$

and results in the following conditions for the matrix:

$$\sigma_{33}^{II}\left(r\ ,\ \ell_{II}\right) = -\,\sigma_{33}^{II}\left(r\ ,\ -\ \ell_{II}\right) = 0 \tag{30}$$

$$\sigma_{32}^{II}\left(r\ ,\ \ell_{II}\right) = -\,\sigma_{32}^{II}\left(r\ ,\ -\ \ell_{II}\right) = 0 \tag{31}$$

$$\sigma_{31}^{II}\left(r\ ,\ \ell_{II}\right) = -\,\sigma_{31}^{II}\left(r\ ,\ -\ \ell_{II}\right) = 0 \tag{32}$$

Applying equation (7) again yields

$$D_{33}^{I}\left(r\ ,\ \pm\ell_I\right) + \frac{\nu^I}{1 - 2\nu^I}\, D_{KK}^{I}\left(r\ ,\ \pm\ell_I\right) = 0 \tag{33}$$

$$D_{32}^{I}\left(r\ ,\ \pm\ell_I\right) = 0 \tag{34}$$

$$D_{31}^{I}\left(r\ ,\ \pm\ell_I\right) = 0 \tag{35}$$

$$D_{33}^{II}\left(r\,,\,\pm\ell_{II}\right) + \frac{\nu^{II}}{1-2\nu^{II}}\,D_{KK}^{II}\left(r\,,\,\pm\ell_{II}\right) = 0 \qquad (36)$$

$$D_{33}^{II}\left(r\,,\,\pm\ell_{II}\right) = 0 \qquad (37)$$

$$D_{31}^{II}\left(r\,,\,\pm\ell_{II}\right) = 0 \qquad (38)$$

and with equations (1) through (6),

$$\frac{\partial\xi_3^I}{\partial z}\left(r\,,\,\pm\ell_I\right) + \frac{\nu^I}{1-2\nu^I}\left\{\frac{1}{r}\,\frac{\partial\, r\xi_1^I}{\partial r}\left(r\,,\,\pm\ell_I\right) + \frac{\partial\xi_3^I}{\partial z}\left(r\,,\,\pm\ell_I\right)\right\} = 0 \qquad (39)$$

$$\frac{\partial\xi_2^I}{\partial z}\left(r\,,\,\pm\ell_I\right) = 0 \qquad (40)$$

$$\frac{\partial\xi_3^I}{\partial r}\left(r\,,\,\pm\ell_I\right) + \frac{\partial\xi_1^I}{\partial z}\left(r\,,\,\pm\ell_I\right) = 0 \qquad (41)$$

$$\frac{\partial\xi_3^{II}}{\partial z}\left(r\,,\,\pm\ell_{II}\right) + \frac{\nu^{II}}{1-2\nu^{II}}\left\{\frac{1}{r}\,\frac{\partial\, r\xi_1^{II}}{\partial r}\left(r\,,\,\ell_{II}\right) + \frac{\partial\xi_3^{II}}{\partial z}\left(r\,,\,\pm\ell_{II}\right)\right\} = 0 \qquad (42)$$

$$\frac{\partial \xi_2^{II}}{\partial z}\left|r \, , \, \pm \ell_{II}\right| = 0 \qquad (43)$$

$$\frac{\partial \xi_3^{II}}{\partial r}\left|r \, , \, \pm \ell_{II}\right| + \frac{\partial \xi_1^{II}}{\partial z}\left|r \, , \, \pm \ell_{II}\right| = 0 \qquad (44)$$

Equations (33) through (44) are twelve boundary conditions relating only to the finite-length fiber. These conditions must be satisfied by the solutions of the differential equations. To define the problem for the infinite fiber, the length ℓ must be assumed to approach infinite, which is a separate mathematical problem.

For reasons of symmetry, the following conditions are valid:

$$\begin{aligned}
\xi_1^{\substack{I\\II}}(r\, ,\, z) &= \xi_1^{\substack{I\\II}}(r\, ,\, -z) \\
\xi_2^{\substack{I\\II}}(r\, ,\, z) &= -\xi_2^{\substack{I\\II}}(r\, ,\, -z) \qquad (45) \\
\xi_3^{\substack{I\\II}}(r\, ,\, z) &= -\xi_3^{\substack{I\\II}}(r\, ,\, -z)
\end{aligned}$$

Since in the plane $z = 0$, equation (45) must be satisfied, ξ_1 can be developed and represented by Taylor's series.

$$\begin{aligned}
\xi^{I,II}(r\, ,\, z) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\partial^n}{\partial z^n} \xi_1^{I,II}(r\, ,\, 0) \\
&= \pm \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \frac{\partial^n}{\partial z^n} \xi_1^{I,II}(r\, ,\, 0)
\end{aligned}$$

or

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left[z^n \mp (-z)^n \right] \frac{\partial^n}{\partial z^n} \xi_i^{I,II} \quad (r , 0) = 0 \tag{46}$$

From equation (46), it follows that for $i = 1$, all uneven partial derivatives with respect to z at the location $(r , 0)$ must vanish identically. All even partial derivatives with respect to z vanish for $i = 2$ and 3, so that

$$\frac{\partial^{2n+1}}{\partial z^{2n+1}} \xi_1^{I,II} \quad (r , 0) = 0$$

and (47)

$$\frac{\partial^{2n}}{\partial z^{2n}} \xi_{2,3}^{I,II} \quad (r , 0) = 0$$

The solutions of equations (8) through (10) must satisfy equation (47).

A third set of conditions can be derived from the fact that on the fiber matrix interface, two neighboring particles of the two materials must remain together during the distortion displacements. First, assume that both the fiber and the matrix are made of an homogeneous material. Secondly, assume that the shrinkage of a cubic particle of the length 1 is $1 - \beta^{I}$ and $1 - \beta^{II}$ respectively. Third, assume that $\beta^{II} > \beta^{I}$. Then the fiber will also be compressed in all directions. As a result, stresses will develop in both materials. Before the stresses develop, the common dimensions are a_o, b_o, and ℓ_o. If the fiber were not present, the inner radius of the resin would shrink freely until it would assume the dimension

$$a^{II} = a_o \left(1 - \beta^{II}\right) \tag{48}$$

Also, the outer radius of the resin reduces

$$b^{II} = b_o \left(1 - \beta^{II}\right) \tag{49}$$

and the length becomes

$$\ell^{II} = \ell_o \left(1 - \beta^{II}\right) \tag{50}$$

A particle in the matrix with the original coordinates

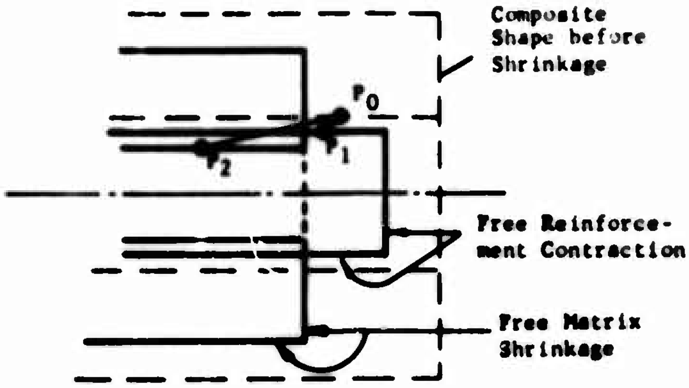
$$r_o \left(1 - \beta^{II}\right) \quad , \quad \varphi_o \quad , \quad z_o \left(1 - \beta^{II}\right)$$

Similar results are obtained for the coordinates of the reinforcement:

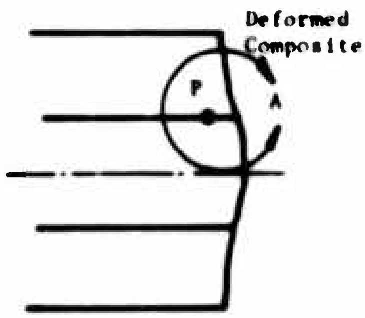
$$a^{I} = a_o \left(1 - \beta^{I}\right) \tag{51}$$

$$\ell^{I} = \ell_o \left(1 - \beta^{I}\right) \tag{52}$$

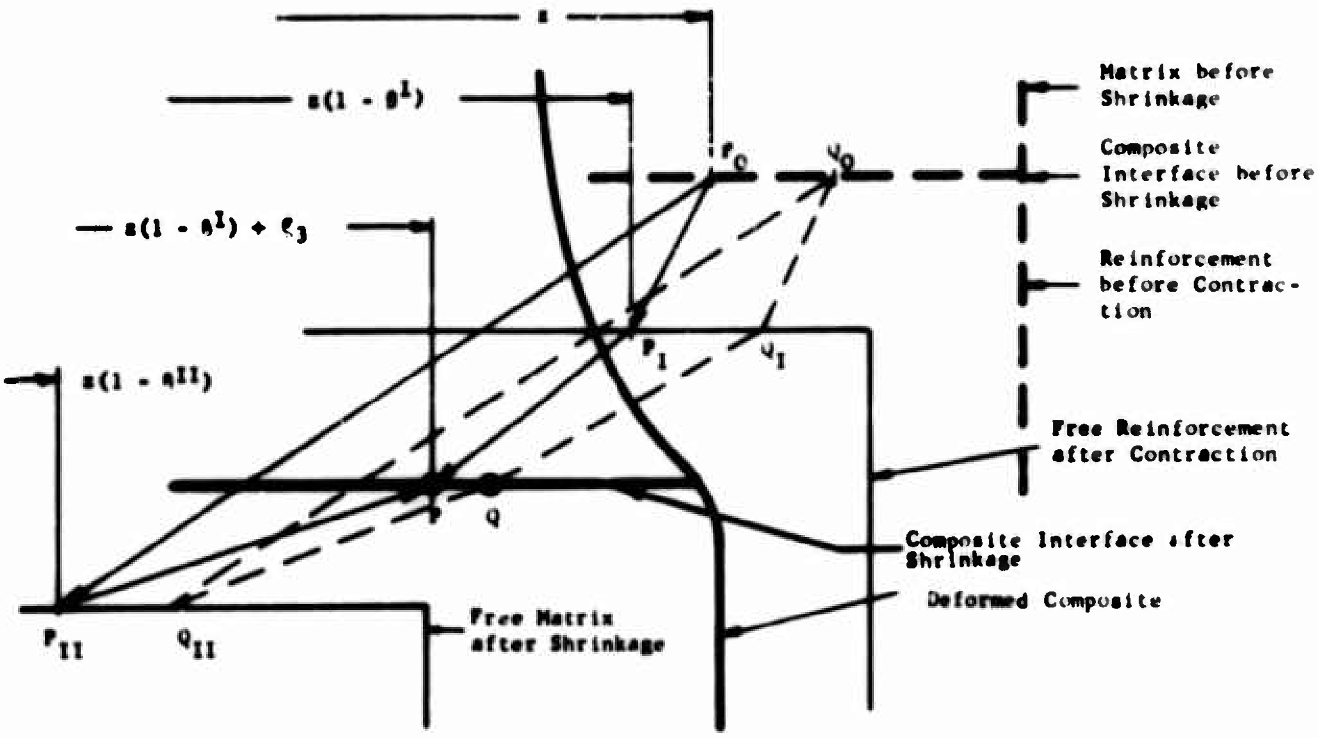
A glass cylinder in a hollow resin cylinder is depicted in Figure 2 in order to illustrate the distortions caused by the difference in shrinkage and thermal expansion of the two materials. Each material by itself would shrink to the cylinder size indicated by the thin lines, but is restricted in this case by the presence of the other material. Since material particles must stay in contact in the interface, the point P_o (in the state without stress) settle at the common location P instead of moving to the locations P_I (for the reinforcement) and P_{II} (for the matrix).



(a) Diagram Depicting Components before and after Independent Shrinkage



(b) Composite Shape Resulting from Combined Shrinkage



(c) View A (Enlarged) Showing Combined Shrinkage in Relation to Independent Shrinkage of Both Materials

Figure 2. Reinforcement Cylinder Embedded in Hollow Matrix Cylinder

The total distortion $\overrightarrow{P_o P}$ can be described in two ways:

$$\overrightarrow{P_o P} = \overrightarrow{P_o P_I} + \overrightarrow{P_I P} \qquad (53)$$

and

$$\overrightarrow{P_o P} = \overrightarrow{P_o P_{II}} + \overrightarrow{P_{II} P} \qquad (54)$$

$\overrightarrow{P_I P}$ is the distortion of the reinforcement cylinder as in the points

$$P_I \left(a_o \left(1 - \beta^I \right) , z \left(1 - \beta^I \right) \right)$$

Therefore,

$$\overrightarrow{P_I P} = \xi_i^I \left(a_o \left(1 - \beta^I \right) , z \left(1 - \beta^I \right) \right) \qquad (55)$$

Similarly, distortion for the matrix is

$$\overrightarrow{P_{II} P} = \xi_i^{II} \left(a_o \left(1 - \beta^{II} \right) , z \left(1 - \beta^{II} \right) \right) \qquad (56)$$

The vectors $\overrightarrow{P_o P_I}$ and $\overrightarrow{P_o P_{II}}$ are displacements of P_o a_o , z in the interface of I and II, so that

$$\overrightarrow{P_o P_I} = V_i^I = - e_{1i} a_o \beta^I - e_{3i} z\beta^I \qquad (57)$$

and

$$\overrightarrow{P_o P_{II}} = V_i^{II} = - e_{1i} a_o \beta^{II} - e_{3i} z\beta^{II} \qquad (58)$$

Using equations (53) and (54), and equations (55) through (58), the following relation exists:

$$\overrightarrow{P_I P} - \overrightarrow{P_{II} P} = \overrightarrow{P_o P_{II}} - \overrightarrow{P_o P_I}$$

or

$$\xi_i^I\left[a_o\left(1 - \beta^I\right),\ z\left(1 - \beta^I\right)\right] - \xi_i^{II}\left[a_o\left(1 - \beta^{II}\right),\ z\left(1 - \beta^{II}\right)\right.$$

$$= e_{1i}\, a_o\left(\beta^I - \beta^{II}\right) + e_{3i}\, z\left(\beta^I - \beta^{II}\right) \qquad (59)$$

The subsequent boundary condition equations follow from equation (59):

$$\xi_1^I\left[a_o\left(1 - \beta^I\right),\ z\left(1 - \beta^I\right)\right] - \xi_1^{II}\left[a_o\left(1 - \beta^{II}\right),\ z\left(1 - \beta^{II}\right)\right]$$

$$= a_o\left(\beta^I - \beta^{II}\right) \qquad (60)$$

$$\xi_2^I\left[a_o\left(1 - \beta^I\right),\ z\left(1 - \beta^I\right)\right] - \xi_2^{II}\left[a_o\left(1 - \beta^{II}\right),\ z\left(1 - \beta^{II}\right)\right] = 0 \qquad (61)$$

$$\xi_3^I\left[a_o\left(1 - \beta^I\right),\ z\left(1 - \beta^I\right)\right] - \xi_3^{II}\left[a_o\left(1 - \beta^{II}\right),\ z\left(1 - \beta^{II}\right)\right]$$

$$= z\left(\beta^I - \beta^{II}\right) \qquad (62)$$

Using the transformations

$$r^I = r\left(1 - \beta^I\right) \qquad (63)$$

$$z^{I} = z\left(1 - \beta^{I}\right) \tag{64}$$

in material I, and the transformations

$$r^{II} = r\left(1 - \beta^{II}\right) \tag{65}$$

$$z^{II} = z\left(1 - \beta^{II}\right) \tag{66}$$

in material II, equations (60) and ((then become

$$\xi_1^{I}\left(a^{I}, z^{I}\right) - \xi_1^{II}\left(a^{II}, z^{II}\right) = a^{II} - a^{I} \tag{67}$$

$$\xi_3^{I}\left(a^{I}, z^{I}\right) - \xi_3^{II}\left(a^{II}, z^{II}\right) = z^{II} - z^{I} \tag{68}$$

By taking into account a strip PQ and its elongation or contraction during the process, and considering Figures 1 and 2(c).

$$\overline{PQ} = \overline{P_I Q_I} + \overline{PP_I} + \overline{Q_I Q} \tag{69}$$

But it is

$$\overline{PP_I} = -\xi_1^{I}\left[a_o\left(1 - \beta^{I}\right), z\left(1 - \beta^{I}\right)\right]$$

and

$$\overline{Q_I Q} = \xi_1^{I}\left[a_o\left(1 - \beta^{I}\right), \left(z + \Delta z\right)\left(1 - \beta^{I}\right)\right] =$$

$$= \xi_i^I\left(a_o\left(1-\beta^I\right), z\left(1-\beta^I\right)\right) + \frac{\partial\xi_i^I}{\partial z}\left(a_o\left(1-\beta^I\right), z\left(1-\beta^I\right)\right)\Delta z\left(1-\beta^I\right) \qquad (70)$$

Writing $\overline{PQ} = V_i^I$, then equation (70) becomes

$$V_i^I = \Delta z\left(1-\beta^I\right)\left\{e_{3i} + \frac{\partial\xi_i^I}{\partial z}\left(a_o\left(1-\beta^I\right), z\left(1-\beta^I\right)\right)\right\} \qquad (71)$$

and for the matrix becomes

$$V_i^{II} = \Delta z\left(1-\beta^{II}\right)\left\{e_{3i} + \frac{\partial\xi_i^{II}}{\partial z}\left(a_o\left(1-\beta^{II}\right), z\left(1-\beta^{II}\right)\right)\right\} \qquad (72)$$

Since

$$V_i^I = V_i^{II} \qquad (73)$$

it follows that

$$\left(1-\beta^I\right)\frac{\partial\xi_i^I}{\partial z}\left(a_o\left(1-\beta^I\right), z\left(1-\beta^I\right)\right) -$$

$$\left(1-\beta^{II}\right)\frac{\partial\xi_i^{II}}{\partial z}\left(a_o\left(1-\beta^{II}\right), z\left(1-\beta^{II}\right)\right) = \delta_{3i}\left(\beta^I - \beta^{II}\right) \qquad (74)$$

Equation (74) could also have been obtained by differentiation from equations (60) and (61); this method can be used to verify the procedure.

On the interface of fiber and resin, there are the normal stresses in both materials equal and opposite

$$P_i^{I} = \sigma_{ij}^{I}\left(a_o\left(1 - \beta^{I}\right),\ z\left(1 - \beta^{I}\right)\right)\delta_{1j} \tag{75}$$

$$P_i^{II} = -\sigma_{ij}^{II}\left(a_o\left(1 - \beta^{II}\right),\ z\left(1 - \beta^{II}\right)\right)\delta_{1j} \tag{76}$$

and

$$P_i^{I} = -P_i^{II} \tag{77}$$

Consequently, the stress boundary condition at the interface is

$$\sigma_{i1}^{I}\left(a_o\left(1 - \beta^{I}\right),\ z\left(1 - \beta^{I}\right)\right) - \sigma_{i1}^{II}\left(a_o\left(1 - \beta^{II}\right),\ z\left(1 - \beta^{II}\right)\right) = 0 \tag{78}$$

Expressed in coordinates, equation (78) is

$$\sigma_{11}^{I}\left[a_o\left(1 - \beta^{I}\right),\ z\left(1 - \beta^{I}\right)\right] - \sigma_{11}^{II}\left[a_o\left(1 - \beta^{II}\right),\ z\left(1 - \beta^{II}\right)\right] = 0 \tag{79}$$

$$\sigma_{12}^{I}\left[a_o\left(1 - \beta^{I}\right),\ z\left(1 - \beta^{I}\right)\right] - \sigma_{12}^{II}\left[a_o\left(1 - \beta^{II}\right),\ z\left(1 - \beta^{II}\right)\right] = 0 \tag{80}$$

$$\sigma_{13}^{I}\left[a_o\left(1 - \beta^{I}\right),\ z\left(1 - \beta^{I}\right)\right] - \sigma_{13}^{II}\left[a_o\left(1 - \beta^{II}\right),\ z\left(1 - \beta^{II}\right)\right] = 0 \tag{81}$$

The six boundary conditions on the common surface of the fiber and resin are given in equations (60) through (62) and equations (79) through (81).

Solution of the Differential Equations for a Single Fiber in the Matrix

After having established all boundary conditions, the differential equations (9) through (11) are treated. The differential equations for one single fiber are brought into the forms

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\left(\frac{\partial^2 \xi_1}{\partial z^2} + \frac{\partial^2 \xi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \xi_1}{\partial r} - \frac{1}{r^2}\xi_1\right) = 0 \qquad (82)$$

$$\frac{\partial^2 \xi_2}{\partial r^2} + \frac{1}{r}\frac{\partial \xi_2}{\partial r} - \frac{1}{r^2}\xi_2 + \frac{\partial^2 \xi_2}{\partial z^2} = 0 \qquad (83)$$

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 \xi_3}{\partial z^2} + \frac{\partial^2 \xi_3}{\partial r^2} + \frac{1}{r}\frac{\partial \xi_3}{\partial r}\right) = 0 \qquad (84)$$

First, equation (82) is solved by setting

$$\frac{\partial^2 \xi_1}{\partial z^2} + \frac{\partial^2 \xi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \xi_1}{\partial r} - \frac{1}{r^2}\xi_1 = \phi_1(r, z) \qquad (85)$$

where ϕ must be a solution of

$$\frac{\partial^2 \phi_1}{\partial z^2} + \frac{\partial^2 \phi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_1}{\partial r} - \frac{1}{r^2}\phi_1 = 0 \qquad (86)$$

According to Bernolli, this equation can be solved by setting

$$\phi_1 = R_1(r)\, Z_1(z)$$

and the following ordinary differential equations obtained:

$$\frac{d^2 Z_1}{dz^2} + \lambda^2 Z_1 = 0 \tag{87}$$

$$\frac{d^2 R_1}{dr^2} + \frac{1}{r}\frac{dR_1}{dr} + \left(-\lambda^2 - \frac{1}{r^2}\right) R_1 = 0 \tag{88}$$

with the solutions

$$Z_1(z) = \begin{cases} e^{+i\lambda z} \\ e^{-i\lambda z} \end{cases} \tag{89}$$

and

$$R_1(r) = \begin{cases} J_1(i\lambda r) \\ H_1^{(1)}(i\lambda r) \end{cases} \tag{90}$$

Hence, equation (85) is

$$\frac{\partial^2 \xi_1}{\partial z^2} + \frac{\partial^2 \xi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \xi_1}{\partial r} - \frac{1}{r^2}\xi_1 = R_1(r)\, Z_1(z) \tag{91}$$

Using a procedure similar to that used for equation (86), the following is set

$$\xi_1 = R(r)\, Z(z) \tag{92}$$

and obtained (Ref. 2)

$$\frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{1}{R}\left\{\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{1}{r^2}R\right\} = \frac{R_1(r)}{R(r)}\;\frac{Z_1(z)}{Z(z)} \tag{93}$$

Two separable cases exist when

$$Z_1(z) = \alpha\, Z(z) \tag{94}$$

and when

$$R_1(r) = \beta\, R(r) \tag{95}$$

Both cases are possible solutions and the sum of both is another solution. For equation (94), it follows from equation (93)

$$-\frac{1}{Z}\frac{d^2Z}{dz^2} = \frac{1}{R}\left\{\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{1}{r^2}R - \alpha\, R_1(r)\right\} = k_1^2 \tag{96}$$

from which it follows that

$$\frac{d^2Z}{dz^2} + k_1^2\, Z = 0 \tag{97}$$

and

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(-\,k_1^2 - \frac{1}{r^2}\right)R = \alpha\, R_1(r) \tag{98}$$

Because of (87), (94), and (97), the following exists:

$$k_1 = \lambda \tag{99}$$

and equation (98) is

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(-\lambda^2 - \frac{1}{r^2}\right) R = \alpha R_1(r) \tag{100}$$

$R_1(r)$ in equation (100) is a linear combination of the solutions, defined by equation (90) as Bessel functions. The total solution of differential equation (100) is:

$$R(r) = A_1 J_1(i\lambda r) + B_1 H_1^{(1)}(i\lambda r) + C_1 r J_o(i\lambda r) + D_1 r H_o(i\lambda r) \tag{101}$$

For the first case, or equation (94), the general solutions are

$$\xi_1 = \left[A_1 J_1(i\lambda r) + B_1 H_1^{(1)}(i\lambda r) + C_1 r J_o(i\lambda r) + D_1 r H_o(i\lambda r)\right]\left[e^{i\lambda z} + a_1 e^{-i\lambda z}\right] \tag{102}$$

For the second case, it follows from equations (95) and (93) that

$$\frac{1}{Z}\frac{d^2Z}{dz^2} - \beta \frac{Z_1(z)}{Z} = k_2^2 \tag{103}$$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(k_2^2 - \frac{1}{r^2}\right) R = 0 \tag{104}$$

Because of equation (95), the equations (104) and (88) are identical, thus

$$k_2 = i\lambda \tag{105}$$

and equation (103) becomes

$$\frac{d^2Z}{dz^2} + \lambda^2 Z = \alpha e^{i\lambda z} + \beta e^{-i\lambda z} \tag{106}$$

Two independent solutions of equation (106) are already found in equation (89). The two other solutions can be found by setting

$$Z(z) = \gamma u(z) e^{i\lambda z} + \delta v(z) e^{-i\lambda z} \tag{107}$$

Equation (106) with equation (107) becomes

$$\gamma\left(u^{11} + 2 u^{1}i\lambda - \lambda^2 u\right)e^{+i\lambda z} + \delta\left(v^{11} - 2 v^{1}i\lambda - \lambda^2 v\right)e^{-i\lambda z}$$

$$\equiv \alpha e^{i\lambda z} + \beta e^{-i\lambda z} \tag{108}$$

From equation (108), the following is obtained:

$$u^{11} + 2 u^{1}i\lambda - \lambda^2 u \equiv \frac{\alpha}{\gamma} \tag{109}$$

$$v^{11} - 2 v^{1}i\lambda - \lambda^2 v \equiv \frac{\beta}{\delta} \tag{110}$$

Since α and β are complex, it follows that for the imaginary part

$$u^1 = \frac{1}{2\lambda} \cdot \mathrm{imag}\left(\frac{\alpha}{\gamma}\right) = C_2 \frac{1}{\gamma}$$

$$v^1 = -\frac{1}{2\lambda} \cdot \text{imag}\left(\frac{e}{\delta}\right) = d_2 \frac{1}{\delta}$$

Consequently,

$$u(z) = \frac{C_2}{\gamma} z \tag{111}$$

$$v(z) = \frac{d_2}{\delta} z \tag{112}$$

The real part gives solutions already known, and equation (107) becomes

$$\bar{Z}(z) = C_2 z\, e^{+i\lambda z} + d_2 z\, e^{-i\lambda z} \tag{113}$$

The complete solution of the ordinary differential equation (106) is

$$Z(z) = a_2\, e^{i\lambda z} + b_2\, e^{-i\lambda z} + C_2 z\, e^{i\lambda z} + d_2 z\, e^{-i\lambda z} \tag{114}$$

and therefore the total solution for the second case

$$\bar{\bar{\xi}}_1 = \left(A_2\, J_1(i\lambda r) + B_2\, H_1^{(1)}(i\lambda r)\right)\left[e^{i\lambda z} + a_2\, e^{-i\lambda z} + b_2 z\, e^{i\lambda r} + C_2 z\, e^{i\lambda r}\right] \tag{115}$$

The total solution including both cases is

$$\xi_1 = \bar{\xi}_1 + \bar{\bar{\xi}}_1 \tag{116}$$

or

$$\xi_1\left(r,z\right) = \left[\bar{A}_2 J_1(i\lambda r) + \bar{B}_2 H_1^{(1)}(i\lambda r)\right]\left[e^{i\lambda z} + \bar{a}_2 e^{-i\lambda z}\right] z +$$

$$\left[\bar{A}_1 J_1(i\lambda r) + \bar{B}_1 H_1^{(1)}(i\lambda r) + \bar{C}_1 r J_o(i\lambda r) + \right.$$

$$\left.\bar{D}_1 r H_o^{(1)}(i\lambda r)\right] \cdot \left[e^{i\lambda z} + \bar{a}_1 e^{-i\lambda z}\right] \quad (117)$$

Using the same procedure for the distortion vector component in axial direction, differential equation (84), a solution analog to equation (117) is obtained:

$$\xi_3\left(r,z\right) = \left[\bar{\bar{A}}_2 J_o(i\bar{\bar{\lambda}} r) + \bar{\bar{B}}_2 H_o^{(1)}(i\bar{\bar{\lambda}} r)\right]\left[e^{i\bar{\bar{\lambda}} z} + \bar{\bar{a}}_2 e^{-i\bar{\bar{\lambda}} z}\right] z +$$

$$\left[\bar{\bar{A}}_1 J_o(i\bar{\bar{\lambda}} r) + \bar{\bar{B}}_1 H_1^{(1)}(i\bar{\bar{\lambda}} r) + \bar{\bar{C}}_1 r J_1(i\bar{\bar{\lambda}} r) + \right.$$

$$\left.\bar{\bar{D}}_1 r H_1^{(1)}(i\bar{\bar{\lambda}} r)\right]\left[e^{i\bar{\bar{\lambda}} z} + \bar{\bar{a}}_1 e^{-i\bar{\bar{\lambda}} z}\right] \quad (118)$$

The general solution of equation (83) is similarly obtained; this is a partial differential equation which results in a Bessel equation of first order and a differential equation of exponential functions

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\mu^2 - \frac{1}{r^2}\right) R = 0$$

and

$$\frac{d^2Z}{dz^2} - \mu^2 Z = 0$$

The solution of equation (83) is the sum of the particular solutions; i.e.,

$$\xi_2(r,z) = \left(\bar{\bar{A}}_1 r + \frac{\bar{\bar{B}}_1}{r}\right)\left(1 + \bar{\bar{a}}_1 z\right) + \left(\bar{\bar{A}}_2\, J_1(i\bar{\bar{\lambda}} r) + \bar{\bar{B}}_2\, H_1^{(1)}(i\bar{\bar{\lambda}} r)\right)\left(e^{i\bar{\bar{\lambda}} z} + \bar{\bar{a}}_2\, e^{-i\bar{\bar{\lambda}} z}\right) \qquad (119)$$

The solutions for the distortion vector components in the reinforcement (superscript I) and the matrix (superscript II) are as follows:

In Radial Direction in the Reinforcement:

$$\xi_1^I(r,z) = \sum_{\lambda^I} J_1\left(i\lambda^I r\right)\left(A_1^I \cos \lambda^I z + A_2^I \sin \lambda^I z\right) z + \left(J_1\left(i\lambda^I r\right) + a_3^I r\, J_o\left(i\lambda^I r\right)\right)\left(A_3^I \cos \lambda^I z + A_4^I \sin \lambda^I z\right) \qquad (120)$$

In Radial Direction in the Matrix:

$$\xi_1^{II}(r,z) = \sum_{\lambda^{II}}\left(J_1\left(i\lambda^{II} r\right) + a_1^{II}\, H_1^{(1)}\left(i\lambda^{II} r\right)\right)\left(A_1^{II} \cos \lambda^{II} z +\right.$$

$$+ A_2^{II} \sin \lambda^{II} z\Big) z + \Big(J_1\big(i\lambda^{II} r\big) + a_2^{II} H_1^{(1)}\big(i\lambda^{II} r\big) +$$

$$a_3^{II} r\, J_o\big(i\lambda^{II} r\big) + a_4^{II} r\, H_o^{(1)}\big(i\lambda^{II} r\big)\Big)\Big(A_3^{II} \cos \lambda^{II} z + A_4^{II} \sin \lambda^{II} z\Big) \qquad (121)$$

In Axial Direction in the Reinforcement:

$$\xi_3^{I}(r,z) = \sum_{\mu^I} J_o\big(i\mu^I r\big)\Big(B_1^I \cos \mu^I z + B_2^I \sin \mu^I z\Big) z + \Big(J_o\big(i\mu^I r\big) +$$

$$b_3^I r\, J_1\big(i\mu^I r\big)\Big)\Big(B_3^I \cos \mu^I z + B_4^I \sin \mu^I z\Big) \qquad (122)$$

In Axial Direction in the Matrix:

$$\xi_3^{II}(r,z) = \sum_{\mu^{II}} \Big(J_o\big(i\mu^{II} r\big) + b_1^{II} H_o^{(1)}\big(i\mu^{II} r\big)\Big)\Big(B_1^{II} \cos \mu^{II} z +$$

$$B_2^{II} \sin \mu^{II} z\Big) z + \Big(J_o\big(i\mu^{II} r\big) + b_2^{II} H_o^{(1)}\big(i\mu^{II} r\big) +$$

$$b_3^{II} r\, J_1\big(i\mu^{II} r\big) + b_4^{II} r\, H_1^{(1)}\big(i\mu^{II} r\big)\Big)\Big(B_3^{II} \cos \mu^{II} z +$$

$$B_4^{II} \sin \mu^{II} z\Big) \qquad (123)$$

In Tangential Direction in the Reinforcement:

$$\xi_2^{I}(r,z) = r\Big(C_1^I + C_2^I z\Big) + \sum_{\nu^I} J_1\big(i\nu^I r\big)\Big(C_3^I \cos \nu^I z + C_4^I \sin \nu^I z\Big) \qquad (124)$$

In Tangential Direction in the Matrix:

$$\zeta_2^{II}(r,z) = \left(r + C_1^{II}\frac{1}{r}\right)\left(C_1^{II} + C_2^{II} z\right) + \sum_{\nu^{II}}\left[J_1\left(i\nu^{II} r\right) + \right.$$

$$\left. C_2^{II} H_1^{(1)}\left(i\nu^{II} r\right)\right]\left(C_3^{II} \cos \nu^{II} z + C_4^{II} \sin \nu^{II} z\right) \qquad (125)$$

<u>Application of the Solutions to the Boundary Conditions Equations for the Integration Constants</u>

Primarily, the solutions must satisfy the boundary conditions expressed in equations (60) through (62). The right-hand side of equations (60) and (62) can be expressed by a Fourier series in the domain

$$\ell_o \leq z \leq + \ell_o$$

The following can be stated:

$$1 = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos \frac{n \pi z}{\ell_o} \qquad (126)$$

and

$$z = \frac{2\ell_o}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n \pi z}{\ell_o} \qquad (127)$$

so that equation (60) can be written in the following form:

$$\xi_1^I\left[a_o\left(1 - \beta^I\right), z\left(1 - \beta^I\right)\right] - \xi_1^{II}\left[a_o\left(1 - \beta^{II}\right), z\left(1 - \beta^{II}\right)\right]$$

$$= 2\, a_o\left(\beta^I - \beta^{II}\right) \sum_{n=1}^{\infty} (-1)^{n+1} \cos \frac{n \pi z}{\ell_o} \qquad (128)$$

Also, equation (62) can be written in similar form:

$$\xi_3^I\left[a_o\left(1 - \beta^I\right), z\left(1 - \beta^I\right)\right] - \xi_3^{II}\left[a_o\left(1 - \beta^{II}\right), z\left(1 - \beta^{II}\right)\right]$$

$$= \frac{2\, \ell_o\left(\beta^I - \beta^{II}\right)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n \pi z}{\ell_o} \qquad (129)$$

The following is obtained for equation (128) by introducing the analytical values for the displacements:

$$\sum_{\lambda^I} J_1\left(i\lambda^I a_o\left(1 - \beta^I\right)\right)\left\{A_1^I \cos\left(\lambda^I z\left(1 - \beta^I\right)\right) + \right.$$

$$\left. A_2^I \sin\left(\lambda^I z\left(1 - \beta^I\right)\right)\right\} z\left(1 - \beta^I\right) + \left(J_1\left(i\lambda^I a_o\left(1 - \beta^I\right)\right) + \right.$$

$$a_3^I a_o\left(1 - \beta^I\right) J_o\left(i\lambda^I a_o\left(1 - \beta^I\right)\right)\left\{A_3^I \cos\left(\lambda^I z\left(1 - \beta^I\right) + \right.\right.$$

$$+ A_4^I \sin\left(\lambda^I z\left(1 - \beta^I\right)\right)\Bigg\}\Bigg] - \sum_{\lambda^{II}} \Bigg[\Bigg\{J_1\left(i\lambda^{II} a_o\left(1 - \beta^{II}\right)\right) +$$

$$a_1^{II} H_1^{(1)}\left(i\lambda^{II} a_o\left(1 - \beta^{II}\right)\right)\Bigg\}\Bigg\{A_1^{II} \cos\left(\lambda^{II} z\left(1 - \beta^{II}\right)\right) +$$

$$A_2^{II} \sin\left(\lambda^{II} z\left(1 - \beta^{II}\right)\right)\Bigg\} z\left(1 - \beta^{II}\right) + \Bigg\{J_1\left(i\lambda^{II} a_o\left(1 - \beta^{II}\right)\right) +$$

$$a_2^{II} H_1^{(1)}\left(i\lambda^{II} a_o\left(1 - \beta^{II}\right)\right) + a_3^{II} a_o\left(1 - \beta^{II}\right) J_o\left(i\lambda^{II} a_o\left(1 - \beta^{II}\right)\right) +$$

$$a_4^{II} a_o\left(1 - \beta^{II}\right) H_o^{(1)}\left(i\lambda^{II} a_o\left(1 - \beta^{II}\right)\right)\Bigg\}\Bigg\{A_3^{II} \cos\left[\lambda^{II} z\left(1 - \beta^{II}\right)\right] +$$

$$A_4^{II} \sin\left[\lambda^{II} z\left(1 - \beta^{II}\right)\right]\Bigg\}\Bigg] = 2 a_o\left(\beta^I - \beta^{II}\right) \sum_{n=1}^{\infty} (-1)^{n+1} \cos \frac{n\pi z}{\ell_o} \qquad (130)$$

The following is obtained for the boundary condition expressed by equation (129):

$$\sum_{\mu^I} \Bigg[J_o\left(i\mu^I a_o\left(1 - \beta^I\right)\right)\Bigg\{B_1^I \cos\left(\mu^I z\left(1 - \beta^I\right)\right) +$$

$$B_2^I \sin\left(\mu^I z\left(1 - \beta^I\right)\right)\Bigg\} z\left(1 - \beta^I\right) + \Bigg\{J_o\left(i\mu^I a_o\left(1 - \beta^I\right)\right) +$$

$$b_3^I a_o\left(1 - \beta^I\right) J_1\left(i\mu^I a_o\left(1 - \beta^I\right)\right)\Bigg\} \Bigg\{B_3^I \cos\left(\mu^I z\left(1 - \beta^I\right)\right) +$$

$$+ B_4^I \sin\left(\mu^I z\left(1 - \beta^I\right)\right)\Bigg\}\Bigg] - \sum_{\mu^{II}} \Bigg[\Bigg\{J_o\left(i\mu^{II} a_o\left(1 - \beta^{II}\right)\right) +$$

$$b_1^{II} H_o^{(1)}\left(i\mu^{II} a_o\left(1 - \beta^{II}\right)\right)\Bigg\} \Bigg\{B_1^{II} \cos\left(\mu^{II} z\left(1 - \beta^{II}\right)\right) +$$

$$B_2^{II} \sin\left(\mu^{II} z\left(1 - \beta^{II}\right)\right)\Bigg\} z\left(1 - \beta^I\right) + \Bigg\{J_o\left(i\mu^{II} a_o\left(1 - \beta^{II}\right)\right) +$$

$$b_2^{II} H_o^{(1)}\left(i\mu^{II} a_o\left(1 - \beta^{II}\right)\right) + b_3^{II} a_o\left(1 - \beta^{II}\right) J_1\left(i\mu^{II} a_o\left(1 - \beta^{II}\right)\right) +$$

$$b_4^{II} a_o\left(1 - \beta^{II}\right) H_1^{(1)}\left(i\mu^{II} a_o\left(1 - \beta^{II}\right)\right)\Bigg\} \Bigg\{B_3^{II} \cos\left(\mu^{II} z\left(1 - \beta^{II}\right)\right) +$$

$$B_4^{II} \sin\left(\mu^{II} z\left(1 - \beta^{II}\right)\right)\Bigg\}\Bigg] = \frac{2\ell_o}{\pi}\left(\beta^I - \beta^{II}\right) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi z}{\ell_o} \qquad (131)$$

Equations (130) and (131) are identically satisfied only when

$$A_1^I = A_2^I = A_1^{II} = A_2^{II} = B_1^I = B_2^I = B_1^{II} = B_2^{II} = 0 \qquad (132)$$

Further,

$$A_4^I = A_4^{II} = B_3^I = B_4^{II} = 0 \qquad (133)$$

is also valid.

The following can also be concluded:

$$\lambda^{I}\left(1 - \beta^{I}\right) = \frac{\pi n}{\ell_o} \tag{134}$$

$$\lambda^{II}\left(1 - \beta^{II}\right) = \frac{n\pi}{\ell_o} \tag{135}$$

$$\mu^{I}\left(1 - \beta^{I}\right) = \frac{n\pi}{\ell_o} \tag{136}$$

$$\mu^{II}\left(1 - \beta^{II}\right) = \frac{n\pi}{\ell_o} \tag{137}$$

where $n = 1, 2, 3, 4 - - -$

From this the following eigenvalues will exist.

$$\lambda^{I} = \mu^{I} = \frac{n\pi}{\ell_o\left(1 - \beta^{I}\right)} \tag{138}$$

$$\lambda^{II} = \mu^{II} = \frac{n\pi}{\ell_o\left(1 - \beta^{II}\right)} \tag{139}$$

Then from equation (130),

$$A_3^{I}\left\{J_1\left(in\pi \frac{a_o}{\ell_o}\right) + a_3^{I}\, a_o\left(1 - \beta^{I}\right) J_o\left(in\pi \frac{a_o}{\ell_o}\right)\right\} - A_3^{II}\left\{J_1\left(in\pi \frac{a_o}{\ell_o}\right) +\right.$$

$$\left. a_2^{II}\, H_1^{(1)}\left(in\pi \frac{a_o}{\ell_o}\right) + a_3^{II}\, a_o\left(1 - \beta^{II}\right) J_o\left(in\pi \frac{a_o}{\ell_o}\right) +\right.$$

$$+ a_4^{II}\, a_o\left(1 - \beta^{II}\right) H_o^{(1)}\left(in\pi \frac{a}{\ell_o}\right)\Bigg\} = 2\, a_o\left(\beta^I - \beta^{II}\right) (-1)^{n+1} \qquad (140)$$

the first equation for the integration constants. The second equation for the integration constants follows from equation (131):

$$B_4^I \left\{J_o\left(in\pi \frac{a_o}{\ell_o}\right) + b_3^I\, a_o\left(1 - \beta^I\right) J_1\left(in\pi \frac{a_o}{\ell_o}\right)\right\} - B_4^{II}\left\{J_o\left(in\pi \frac{a_o}{\ell_o}\right) + \right.$$

$$b_2^{II}\, H_o^{(1)}\left(in\pi \frac{a_o}{\ell_o}\right) + b_3^{II}\, a_o\left(1 - \beta^{II}\right) J_1\left(in\pi \frac{a_o}{\ell_o}\right) +$$

$$\left. b_4^{II}\, a_o\left(1 - \beta^{II}\right) H_1^{(1)}\left(in\pi \frac{a_o}{\ell_o}\right)\right\} = \frac{2\,\ell_o}{\pi} \frac{(-1)^{n+1}}{n}\left(\beta^I - \beta^{II}\right) \qquad (141)$$

By considering the boundary condition expressed by equation (61),

$$a_o\left(1 - \beta^I\right)\left(C_1^I + C_2^I\, z\left(1 - \beta^I\right)\right) +$$

$$\sum_{\nu^I} J_1\left(i\nu^I\, a_o\left(1 - \beta^I\right)\right)\left\{C_3^I \cos\left(\nu^I\, z\left(1 - \beta^I\right)\right) + \right.$$

$$\left. C_4^I \sin\left(\nu^I\, z\left(1 - \beta^I\right)\right)\right\} - \left(a_o\left(1 - \beta^{II}\right) + \right.$$

$$\left. C_1^{II} \frac{1}{a_o\left(1 - \beta^{II}\right)}\right)\left(C_1^{II} + C_2^{II}\, z\left(1 - \beta^{II}\right)\right) -$$

$$- \sum_{\nu^{II}} \left(J_1\left(i\nu^{II} a_o\left(1 - \beta^{II}\right) + C_2^{II} H_1^{(1)}\left(i\nu^{II} a_o\left(1 - \beta^{II}\right)\right)\right) \left(C_3^{II} \cos\left(\nu^{II} z\left(1 - \beta^{II}\right)\right) + C_4^{II} \sin\left(\nu^{II} z\left(1 - \beta^{II}\right)\right)\right) = 0 \quad (142)$$

When equation (142) is considered, then, all coefficients from sine and cosine functions from z disappear identically

$$C_1^I = C_2^I = C_3^I = C_4^I = 0 \quad (143)$$

$$C_1^{II} = C_2^{II} = C_3^{II} = C_4^{II} = 0 \quad (144)$$

Equations (120 through (125) can now be written, representing the distortion as follows:

Displacement in the Reinforcement (Radial Direction):

$$s_1^I(r,z) = \sum_{n=1}^{\infty} A_3^I \left\{ J_1\left(i \frac{n\pi}{1 - \beta^I} \frac{r}{\ell_o}\right) + a_3^I r\, J_o\left(i \frac{n\pi}{1 - \beta^I} \frac{r}{\ell_o}\right)\right\} \cos \frac{n\pi z}{\ell_o\left(1 - \beta^I\right)} \quad (145)$$

Displacement in the Matrix (Radial Direction):

$$\xi_1^{II}(r,z) = \sum_{n=1}^{\infty} A_3^{II}\left\{J_1\left(i\frac{n\pi}{1-\beta^{II}}\frac{r}{\ell_o}\right) + a_2^{II}\,H_1^{(1)}\left(i\frac{n\pi}{1-\beta^{II}}\frac{r}{\ell_o}\right) + a_3^{II}r\,J_o\left(i\frac{n\pi}{1-\beta^{II}}\frac{r}{\ell_o}\right) + a_4^{II}r\,H_o^{(1)}\left(i\frac{n\pi}{1-\beta^{II}}\frac{r}{\ell_o}\right)\right\}\cos\frac{n\pi z}{\ell_o\left(1-\beta^{II}\right)} \tag{146}$$

Displacement in the Reinforcement (Tangential Direction):

$$\xi_2^{I}(r,z) = 0 \tag{147}$$

Displacement in the Matrix (Tangential Direction):

$$\xi_2^{II}(r,z) = 0 \tag{148}$$

Displacement in the Reinforcement (Axial Direction):

$$\xi_3^{I}(r,z) = \sum_{n=1}^{\infty} B_4^{I}\left\{J_o\left(i\frac{n\pi}{1-\beta^{I}}\frac{r}{\ell_o}\right) + b_3^{I}r\,J_1\left(i\frac{n\pi}{1-\beta^{I}}\frac{r}{\ell_o}\right)\right\}\sin\frac{n\pi z}{\ell_o\left(1-\beta^{I}\right)} \tag{149}$$

Displacement in the Matrix (Axial Direction):

$$\xi_3^{II}(r,z) = \sum_{n=1}^{\infty} B_4^{II}\left\{J_o\left(i\frac{n\pi}{1-\beta^{II}}\frac{r}{\ell_o}\right) + b_2^{II} H_o^{(1)}\left(i\frac{n\pi}{1-\beta^{II}}\frac{r}{\ell_o}\right) + b_3^{II} r\, J_1\left(i\frac{n\pi}{1-\beta^{II}}\frac{r}{\ell_o}\right) + b_4^{II} r\, H_1^{(1)}\left(i\frac{n\pi}{1-\beta^{II}}\frac{r}{\ell_o}\right)\right\}\sin\frac{n\pi z}{\ell_o\left(1-\beta^{II}\right)} \qquad (150)$$

The boundary conditions expressed by equations (39) and (42) must now is satisfied. The displacement given in equations (145) and (149) are substituted into the boundary condition expressed in equation (39):

$$\frac{1-\nu^I}{1-2\nu^I}\sum_{n=1}^{\infty} B_4^I\left\{J_o\left(\frac{n\pi i}{1-\beta^I}\frac{r}{\ell_o}\right) + b^I r\, J_1\left(i\frac{n\pi}{1-\beta^I}\frac{r}{\ell_o}\right)\right\}\frac{n\pi}{\ell_o\left(1-\beta^I\right)}\cos n\pi +$$

$$\frac{\nu^I}{1-2\nu^I}\frac{1}{r}\sum_{n=1}^{\infty} A_3^I\frac{d}{dr}\left\{r\, J_1\left(i\frac{n\pi r}{\ell_o\left(1-\beta^I\right)}\right) + a_3^I r^2\, J_o\left(i\frac{n\pi}{1-\beta^I}\frac{r}{\ell_o}\right)\right\}\cos n\pi = 0 \qquad (151)$$

Using differentiation formulas of the cylinder functions,

$$\sum_{n=1}^{\infty}(-1)^n \left\{ J_o\left(i \frac{n\pi}{\ell_o\left(1 - \beta^I\right)} \frac{r}{\ell_o}\right)\left[\frac{n\pi}{\ell_o\left(1 - \beta^I\right)} \frac{1 - \nu^I}{1 - 2\nu^I} B_4^I + \right.\right.$$

$$\left. \frac{\nu^I}{1 - 2\nu^I} A_3^I \left(\frac{i n\pi}{\ell_o\left(1 - \beta^I\right)} + 2\, a_3^I\right)\right] +$$

$$\frac{n\pi}{\ell_o\left(1 - \beta^I\right)}\, r\, J_1\left(i \frac{n\pi}{\ell_o\left(1 - \beta^I\right)} \frac{r}{\ell_o}\right)\left[B_4^I\, b^I \frac{1 - \nu^I}{1 - 2\nu^I} - \right.$$

$$\left.\left. \frac{\nu^I}{1 - 2\nu^I}\, i\, A_3^I\, a_3^I\right]\right\} = 0 \qquad (152)$$

In equation (152), all coefficients of the Bessel functions J_o and $r\,J_1$ disappear because the equation must be satisfied for all r in material I (reinforcement); this results in the two subsequent equations:

$$\frac{n\pi}{\ell_o\left(1 - \beta^I\right)}\left(1 - \nu^I\right)B_4^I + \nu^I A_3^I\left(2\, a_3^I + i \frac{n\pi}{\ell_o\left(1 - \beta^I\right)}\right) = 0 \qquad (153)$$

and

$$B_4^I\, b_3^I\left(1 - \nu^I\right) - \nu^I\, i\, A_3^I\, a_3^I = 0 \qquad (154)$$

Equations (153) and (154) represent the third and fourth equations for the integration constants.

By utilizing a similar approach and the conclusion that the expressions must vanish for all r in material II (matrix), the subsequent equations are obtained for the integration constants by introducing equations (146) and (150) into the boundary condition expressed in equation (42):

$$\left(1 - \nu^{II}\right) \frac{n\pi}{\ell_o\left(1 - \beta^{II}\right)} B_4^{II} + \nu^{II} \frac{n\pi}{\ell_o\left(1 - \beta^{II}\right)} i\, A_3^{II} + 2\nu^{II} a_3^{II} A_3^{II} = 0 \qquad (155)$$

$$\left(1 - \nu^{II}\right) b_3^{II} B_4^{II} - \nu^{II} i\, A_3^{II} a_3^{II} = 0 \qquad (156)$$

$$\left(1 - \nu^{II}\right) \frac{n\pi}{\ell_o\left(1 - \beta^{II}\right)} b_2^{II} B_4^{II} + \nu^{II} \frac{n\pi}{\ell_o\left(1 - \beta^{II}\right)} i\, a_2^{II} A_3^{II} + 2\nu^{II} a_4^{II} A_3^{II} = 0 \qquad (157)$$

$$\left(1 - \nu^{II}\right) b_4^{II} B_4^{II} - \nu^{II} i\, a_3^{II} A_3^{II} = 0 \qquad (158)$$

The end conditions of a finite fiber resulted in equations (153) through (158). Note that the boundary conditions expressed by equations (41) and (44) are automatically satisfied and hence do not furnish equations for integration constants.

Two additional equations for the integration constants are obtained when the boundary condition equations (22) to (24), which states that the stress vanishes on the surface of the resin, are satisfied.

Introduction of the corresponding solutions for distortion from equations (146) and (150) into equation (22) yields a mathematical expression containing terms, with a common coefficient

$$\cos n\pi \frac{z}{\ell_o}$$

Since the expression must be zero for all values of z, and since the cosine is not of this nature, the complete expression must vanish. Use of this conclusion results in a ninth equation for the integration constants

$$
\begin{aligned}
A_3^{II}\Bigg\{ & -\frac{\ell_o}{b_o}\frac{1-2\nu^{II}}{\left(1-\beta^{II}\right)} J_1\left(i n\pi \frac{b_o}{\ell_o}\right) + i n\pi \frac{1-\nu^{II}}{1-\beta^{II}} J_o\left(i n\pi \frac{b_o}{\ell_o}\right) + \\
& a_2^{II}\left[-\frac{\ell_o}{b_o}\frac{1-2\nu^{II}}{\left(1-\beta^{II}\right)} H_1^{(1)}\left(i n\pi \frac{b_o}{\ell_o}\right) + \right. \\
& \left. i n\pi \frac{1-\nu^{II}}{1-\beta^{II}} H_o^{(1)}\left(i n\pi \frac{b_o}{\ell_o}\right)\right] - a_3^{II}\left[\ell_o J_o\left(i n\pi \frac{b_o}{\ell_o}\right) + \right. \\
& \left. i\left(1-\nu^{II}\right) b_o n\pi J_1\left(i n\pi \frac{b_o}{\ell_o}\right)\right] - a_4^{II}\left[\ell_o H_o^{(1)}\left(i n\pi \frac{b_o}{\ell_o}\right) + \right. \\
& \left. i\left(1-\nu^{II}\right) b_o n\pi H_1^{(1)}\left(i n\pi \frac{b_o}{\ell_o}\right)\right]\Bigg\} + \\
& \nu^{II}\frac{n\pi}{1-\beta^{II}} B_4^{II}\left\{J_o\left(i n\pi \frac{b_o}{\ell_o}\right) + b_2^{II} H_o^{(1)}\left(i n\pi \frac{b_o}{\ell_o}\right) + \right.
\end{aligned}
$$

$$+ b_3^{II} b_o \left(1 - \beta^{II}\right) J_1\left(in\pi \frac{b_o}{\ell_o}\right) +$$

$$b_4^{II} b_o \left(1 - \beta^{II}\right) H_1^{(1)}\left(in\pi \frac{b_o}{\ell_o}\right)\Bigg\} = 0 \qquad (159)$$

A similar equation is obtained, utilizing similar conclusions, by using the boundary condition expressed in equation (14) and the solutions from equations (146) and (150). As in the former cases, the coefficients of the trigonometric functions must disappear, and the following equation is obtained:

$$i B_4^{II} \Bigg\{J_1\left(in\pi \frac{b_o}{\ell_o}\right) + b_2^{II} H_1^{(1)}\left(in\pi \frac{b_o}{\ell_o}\right) - b_3^{II} J_o\left(in\pi \frac{b_o}{\ell_o}\right) -$$

$$b_4^{II} H_o^{(1)}\left(in\pi \frac{b_o}{\ell_o}\right)\Bigg\} + A_3^{II} \Bigg\{J_1\left(in\pi \frac{b_o}{\ell_o}\right) +$$

$$a_2^{II} H_1^{(1)}\left(in\pi \frac{b_o}{\ell_o}\right) + a_3^{II} b_o \left(1 - \beta^{II}\right) J_o\left(in\pi \frac{b_o}{\ell_o}\right) +$$

$$a_4^{II} b_o \left(1 - \beta^{II}\right) H_o^{(1)}\left(in\pi \frac{b_o}{\ell_o}\right)\Bigg\} = 0 \qquad (160)$$

Two additional equations for the coefficient are obtained from the boundary conditions given in equations (79) and (81). Substituting σ_{11}^{I} and σ_{11}^{II} into equation (79) yields an equation containing cylindrical functions multiplied by $\cos n\pi \frac{z}{\ell_o}$. Since the equation base must be satisfied for any

$$\cos n\pi \frac{z}{\ell_o}$$

the coefficients of this expression must vanish, and the eleventh equation for the coefficients is obtained. Equations (145), (146), (149), and (150 are utilized.

$$
\begin{aligned}
&\frac{E^{I}}{\left(1+\nu^{I}\right)\left(1-2\nu^{I}\right)}\left\{A_3^{I}\left[-\frac{1-2\nu^{I}}{a_o\left(1-\beta^{I}\right)}\,J_1\left(in\pi\,\frac{a_o}{\ell_o}\right)+\right.\right.\\
&\qquad i\,\frac{n\pi\left(1-\nu^{I}\right)}{\ell_o\left(1-\beta^{I}\right)}\,J_o\left(in\pi\,\frac{a_o}{\ell_o}\right)+a_3^{I}\left(\left(1-2\nu^{I}\right)\,J_o\left(in\pi\,\frac{a_o}{\ell_o}\right)-\right.\\
&\qquad\qquad \left.\left. in\pi\,\frac{a_o}{\ell_o}\left(1-\nu^{I}\right)\,J_1\left(in\pi\,\frac{a_o}{\ell_o}\right)\right)\right]+\\
&\qquad \left.\nu^{I}\,\frac{n\pi}{\ell_o\left(1-\beta^{I}\right)}\,B_4^{I}\left[J_o\left(in\pi\,\frac{a_o}{\ell_o}\right)+b_3^{I}\,J_1\left(in\pi\,\frac{a_o}{\ell_o}\right)\right]\right\}-\\
&\frac{E^{II}}{\left(1+\nu^{II}\right)\left(1-2\nu^{II}\right)}\left\{A_3^{II}\left[-\frac{1-2\nu^{II}}{a_o\left(1-\beta^{II}\right)}\,J_1\left(in\pi\,\frac{a_o}{\ell_o}\right)+\right.\right.\\
&i\,\frac{n\pi\left(1-\nu^{II}\right)}{\ell_o\left(1-\beta^{II}\right)}\,J_o\left(in\pi\,\frac{a_o}{\ell_o}\right)+a_2^{II}\left(-\frac{1-2\nu^{II}}{a_o\left(1-\beta^{II}\right)}\,H_1^{(1)}\left(in\pi\,\frac{a_o}{\ell_o}\right)+\right.\\
&\qquad \left. i\,\frac{n\pi\left(1-\nu^{II}\right)}{\ell_o\left(1-\beta^{II}\right)}\,H_o^{(1)}\left(in\pi\,\frac{a_o}{\ell_o}\right)\right)+a_3^{II}\left(J_o\left(in\pi\,\frac{a_o}{\ell_o}\right)-\right.\\
&\qquad \left. in\pi\,\frac{a_o}{\ell_o}\left(1-\nu^{II}\right)\,J_1\left(in\pi\,\frac{a_o}{\ell_o}\right)\right)+a_4^{II}\left(H_o^{(1)}\left(in\pi\,\frac{a_o}{\ell_o}\right)-\right.
\end{aligned}
$$

$$- i n\pi \frac{a_o}{\ell_o} \left(1 - \nu^{II}\right) H_1^{(1)}\left(i n\pi \frac{a_o}{\ell_o}\right)\Bigg) +$$

$$\nu^{II} \frac{n\pi}{\ell_o\left(1 - \beta^{II}\right)} B_4^{II}\left(J_o\left(i n\pi \frac{a_o}{\ell_o}\right) + b_2^{II} H_o^{(1)}\left(i n\pi \frac{a_o}{\ell_o}\right) +\right.$$

$$\left. b_3^{II} a_o\left(1 - \beta^{II}\right) J_1\left(i n\pi \frac{a_o}{\ell_o}\right) + b_4^{II} a_o\left(1 - \beta^{II}\right) H_1^{(1)}\left(i n\pi \frac{a_o}{\ell_o}\right)\right)\Bigg\} = 0 \qquad (161)$$

Using the boundary condition expressed in equation (81), the following twelfth equation for the integration constants is obtained after introducing the functions for σ_{13}^{I} and σ_{13}^{II} :

$$\frac{E^I}{1 + \nu^I} \frac{1}{1 - \beta^I} \Bigg\{ i B_4^I\left[- J_1\left(i n\pi \frac{a_o}{\ell_o}\right) + b_3^I J_o\left(i n\pi \frac{a_o}{\ell_o}\right)\right] -$$

$$A_3^I\left[J_1\left(i n\pi \frac{a_o}{\ell_o}\right) + a_3^I a_o\left(1 - \beta^I\right) J_o\left(i n\pi \frac{a_o}{\ell_o}\right)\right]\Bigg\} -$$

$$\frac{E^{II}}{1 + \nu^{II}} \frac{1}{1 - \beta^{II}} \Bigg\{ i B_4^{II}\left[- J_1\left(i n\pi \frac{a_o}{\ell_o}\right) - b_2^{II} H_1^{(1)}\left(i n\pi \frac{a_o}{\ell_o}\right) +\right.$$

$$\left. b_3^{II} J_o\left(i n\pi \frac{a_o}{\ell_o}\right) + b_4^{II} H_o^{(1)}\left(i n\pi \frac{a_o}{\ell_o}\right)\right] - A_3^{II}\left[J_1\left(i n\pi \frac{a_o}{\ell_o}\right) +\right.$$

$$a_2^{II} H_1^{(1)}\left(i n\pi \frac{a_o}{\ell_o}\right) + a_3^{II} a_o\left(1 - \beta^{II}\right) J_o\left(i n\pi \frac{a_o}{\ell_o}\right) +$$

$$+ a_4^{II} a_o\left(1 - \beta^{II}\right) H_o^{(1)}\left(i n\pi \frac{a_o}{\ell_o}\right)\Bigg\} = 0 \qquad (162)$$

The twelve equations - equations (140), (141), (153), (154), and (155) through (162) - exhaust all technically meaningful boundary conditions. The coefficients to be determined by these equations are

$$A_3^I \; ; \; A_3^I a_3^I \; ; \; A_3^{II} \; ; \; A_3^{II} a_2^{II} \; ; \; A_3^{II} a_3^{II} \; ; \; A_3^{II} a_4^{II}$$

$$B_4^I \; ; \; B_4^I b_3^I \; ; \; B_4^{II} \; ; \; B_4^{II} b_2^{II} \; ; \; B_4^{II} b_3^{II} \; ; \; B_4^{II} b_4^{II}$$

The twelve equations are inhomogeneous and linear.

<u>At this point, the physical problem of one finite-length fiber in a matrix is solved</u>.

The determination of the constant is a problem of pure mathematical analysis. Further interrelation to the physical problem is unnecessary.

<u>Determination of the Integration Constants</u>

Because of equations (153) through (158), the coefficients

$$B_4^I \; , \; B_4^I b_3^I \; , \; B_4^{II} \; , \; B_4^{II} b_3^{II} \; , \; B_4^{II} b_2^{II} \; , \text{ and } \; B_4^{II} b_4^{II}$$

can be expressed by the corresponding A coefficients. Equations (141), (159), (160), (161), and (162) can be transformed by this relation. Equation (140) can be taken without transformation and restated as

$$A_3^I \left\{ J_1\left(i n\pi \frac{a_o}{\ell_o}\right) + a_3^I a_o\left(1 - \beta^I\right) J_o\left(i n\pi \frac{a_o}{\ell_o}\right)\right\} - A_3^{II} \left\{ J_1\left(i n\pi \frac{a_o}{\ell_o}\right) + \right.$$

$$+ a_2^{II} H_1^{(1)}\left(in\pi \frac{a_o}{\ell_o}\right) + a_3^{II} a_o\left(1 - \beta^{II}\right) J_o\left(in\pi \frac{a_o}{\ell_o}\right) +$$

$$a_4^{II} a_o\left(1 - \beta^{II}\right) H_o^{(1)}\left(in\pi \frac{a_o}{\ell_o}\right)\Bigg\}$$

$$= 2 a_o\left(\beta^{I} - \beta^{II}\right)(- 1)^{n+1} \qquad (163)$$

Only the results of an extensive analysis are given in the following five equations.

Equation (141) becomes

$$\frac{\nu^{I}}{1 - \nu^{I}}\left\{i A_3^{I} J_o\left(in\pi \frac{a_o}{\ell_o}\right) + A_3^{I} a_3^{I} \left(1 - \beta^{I}\right)\ell_o \left[\frac{2}{n\pi} J_o\left(in\pi \frac{a_o}{\ell_o}\right) +\right.\right.$$

$$\left.\left.\frac{a_o}{\ell_o} i J_1\left(in\pi \frac{a_o}{\ell_o}\right)\right]\right\} + \frac{\nu^{II}}{1 - \nu^{II}}\left\{i A_3^{II} J_o\left(in\pi \frac{a_o}{\ell_o}\right) +\right.$$

$$A_3^{II} a_2^{II} i H_o\left(in\pi \frac{a_o}{\ell_o}\right) + A_3^{II} a_3^{II} \ell_o\left(1 - \beta^{II}\right)\left[\frac{2}{n\pi} J_o\left(in\pi \frac{a_o}{\ell_o}\right) -\right.$$

$$\left.\frac{a_o}{\ell_o} i J_1\left(in\pi \frac{a_o}{\ell_o}\right)\right] - i A_3^{II} a_4^{II} \ell_o\left(1 - \beta^{II}\right)\left[\frac{2}{n\pi} i H_o^{(1)}\left(in\pi \frac{a_o}{\ell_o}\right) +\right.$$

$$\left.\left.\frac{a_o}{\ell_o} H_1^{(1)}\left(in\pi \frac{a_o}{\ell_o}\right)\right]\right\}$$

$$= \frac{2 \ell_o}{\pi} \frac{(- 1)^{n+1}}{n}\left(\beta^{I} - \beta^{II}\right) \qquad (164)$$

Another equation is obtained through modification of equation (159) by introducing equations (155) through (158)

$$
\begin{aligned}
& i\, A_3^{II} \left\{ \frac{\ell_o}{b_o}\, \frac{1 - 2\nu^{II}}{1 - \beta^{II}}\, i\, J_1\!\left(in\pi \frac{b_o}{\ell_o}\right) + \right. \\
& \qquad \left. \frac{n\pi}{1 - \beta^{II}}\, \frac{1 - 2\nu^{II}}{1 - \nu^{II}}\, J_o\!\left(in\pi \frac{b_o}{\ell_o}\right) \right\} + \\
& A_3^{II}\, a_2^{II} \left\{ - \frac{\ell_o}{b_o}\, \frac{1 - 2\nu^{II}}{1 - \beta^{II}}\, H_1^{(1)}\!\left(in\pi \frac{b_o}{\ell_o}\right) + \right. \\
& \qquad \left. \frac{n\pi}{1 - \beta^{II}}\, \frac{1 - 2\nu^{II}}{1 - \beta^{II}}\, i\, H_o^{(1)}\!\left(in\pi \frac{b_o}{\ell_o}\right) \right\} - \\
& A_3^{II}\, a_3^{II}\, \ell_o \left\{ \frac{2\left(\nu^{II}\right)^2 - \nu^{II} + 1}{1 - \nu^{II}}\, J_o\!\left(in\pi \frac{b_o}{\ell_o}\right) + \right. \\
& \qquad \left. n\pi \frac{b_o}{\ell_o}\, \frac{1 - 2\nu^{II}}{1 - \nu^{II}}\, i\, J_1\!\left(in\pi \frac{b_o}{\ell_o}\right) \right\} + \\
& i\, A_3^{II}\, a_4^{II}\, \ell_o \left\{ \frac{2\left(\nu^{II}\right)^2 - \nu^{II} + 1}{1 - \nu^{II}}\, i\, H_o\!\left(in\pi \frac{b_o}{\ell_o}\right) - \right. \\
& \qquad \left. n\pi \frac{b_o}{\ell_o}\, \frac{1 - 2\nu^{II}}{1 - \nu^{II}}\, H_1^{(1)}\!\left(in\pi \frac{b_o}{\ell_o}\right) \right\} = 0 \qquad (165)
\end{aligned}
$$

Next, equation (160) is transformed by introducing equations (155) to (158). Consequently,

$$- i A_3^{II} i J_1\left(in\pi \frac{b_o}{\ell_o}\right) + A_3^{II} a_2^{II} H_1^{(1)}\left(in\pi \frac{b_o}{\ell_o}\right) +$$

$$A_3^{II} a_3^{II} \ell_o\left(1 - \beta^{II}\right)\left[\frac{b_o}{\ell_o} J_o\left(in\pi \frac{b_o}{\ell_o}\right) - \frac{2\nu^{II}}{n\pi} i J_1\left(in\pi \frac{b_o}{\ell_o}\right)\right] -$$

$$i A_3^{II} a_4^{II} \ell_o\left(1 - \beta^{II}\right)\left[\frac{b_o}{\ell_o} i H_o^{(1)}\left(in\pi \frac{b_o}{\ell_o}\right) + \frac{2\nu^{II}}{n\pi} H_1^{(1)}\left(in\pi \frac{b_o}{\ell_o}\right)\right] = 0 \quad (166)$$

Equation (161) becomes, with equations (141), (153), and (154)

$$i A_3^{I} \left\{\frac{E^{I}}{\left(1 + \nu^{I}\right)\left(1 - 2\nu^{I}\right)} \frac{1}{1 - \beta^{I}} \left(\frac{1 - 2\nu^{I} + \left(\nu^{I}\right)^2}{1 - \nu^{I}} J_o\left(in\pi \frac{a_o}{\ell_o}\right) +\right.\right.$$

$$\left.\frac{\ell_o}{a_o} \frac{1 - 2\nu^{I}}{n\pi} i J_1\left(in\pi \frac{a_o}{\ell_o}\right)\right) -$$

$$\left.\frac{E^{II} \nu^{I} \nu^{II}}{\left(1 + \nu^{II}\right)\left(1 - 2\nu^{II}\right)} \frac{1}{\left(1 - \nu^{I}\right)\left(1 - \beta^{II}\right)} J_o\left(in\pi \frac{a_o}{\ell_o}\right)\right\} +$$

$$A_3^{I} a_3^{I} \ell_o \left\{\frac{E^{I}}{\left(1 + \nu^{I}\right)\left(1 - 2\nu^{I}\right)} \frac{1}{n\pi} \frac{1 - 3\nu^{I} + 4\left(\nu^{I}\right)^2}{1 - \nu^{I}} J_o\left(in\pi \frac{a_o}{\ell_o}\right) -\right.$$

$$\left.\frac{a_o}{\ell_o} \frac{1 - 2\nu^{I}}{1 - \nu^{I}} i J_1\left(in\pi \frac{a_o}{\ell_o}\right)\right] -$$

$$- \frac{E^{II} \nu^{I} \nu^{II}}{\left(1 + \nu^{II}\right)\left(1 - 2\nu^{II}\right)} \frac{1 - \beta^{I}}{1 - \beta^{II}} \left[\frac{2}{n\pi} J_o\left(in\pi \frac{a_o}{\ell_o}\right) + \right.$$

$$\left. \left. \frac{a_o}{\ell_o} i J_1\left(in\pi \frac{a_o}{\ell_o}\right)\right]\right\} - i A_3^{II} \frac{E^{II} \; 1 - \nu^{II}}{\left(1 + \nu^{II}\right)\left(1 - 2\nu^{II}\right)} \frac{1}{1 - \beta^{II}} \left\{ J_o\left(in\pi \frac{a_o}{\ell_o}\right) + \right.$$

$$\left. \frac{\ell_o}{a_o} \frac{1 - 2\nu^{II}}{1 - \nu^{II}} i J_1\left(in\pi \frac{a_o}{\ell_o}\right)\right\} - A_3^{II} a_2^{II} \frac{E^{II} \; 1 - \nu^{II}}{\left(1 + \nu^{II}\right)\left(1 - 2\nu^{II}\right)} \frac{1}{1 - \beta^{II}} -$$

$$\left\{ i H_o^{(1)}\left(in\pi \frac{a_o}{\ell_o}\right) - \frac{\ell_o}{a_o} \frac{1 - 2\nu^{II}}{1 - \nu^{II}} H_1^{(1)}\left(in\pi \frac{a_o}{\ell_o}\right)\right\} -$$

$$A_3^{II} a_3^{II} \frac{E^{II} \ell_o}{\left(1 + \nu^{II}\right)\left(1 - 2\nu^{II}\right)} \left\{ \frac{1}{n\pi} J_o\left(in\pi \frac{a_o}{\ell_o}\right) - \right.$$

$$\left. \frac{a_o}{\ell_o} \left(1 - \nu^{II}\right) i J_1\left(in\pi \frac{a_o}{\ell_o}\right)\right\} +$$

$$i A_3^{II} a_4^{II} \frac{E^{II} \ell_o}{\left(1 + \nu^{II}\right)\left(1 - 2\nu^{II}\right)} \left\{ + \frac{1}{n\pi} i H_o^{(1)}\left(in\pi \frac{a_o}{\ell_o}\right) + \right.$$

$$\left. \frac{a_o}{\ell_o} \left(1 - \nu^{II}\right) H_1^{(1)}\left(in\pi \frac{a_o}{\ell_o}\right)\right\}$$

$$= \frac{2 E^{II} \nu^{II} \ell_o}{n\pi\left(1 + \nu^{II}\right)\left(1 - 2\nu^{II}\right)} \frac{\beta^{I} - \beta^{II}}{1 - \beta^{II}} \qquad (167)$$

Finally, equation (162) must be transformed by expressing all B's by A's by means of equations (153) through (158).

$$i\,A_3^I \frac{E^I}{\left(1+\nu^I\right)\left(1-2\nu^I\right)} \frac{1}{1-\beta^I}\left\{\frac{1-2\nu^I}{1-\nu^I} J_o\left(in\pi \frac{a_o}{\ell_o}\right) + \right.$$

$$\left. \frac{\ell_o}{a_o} \frac{1-2\nu^I}{\pi n}\, i\, J_1\left(in\pi \frac{a_o}{\ell_o}\right)\right\} +$$

$$A_3^I\, a_3^I \frac{E^I\,\ell_o}{\left(1+\nu^I\right)\left(1-2\nu^I\right)}\left\{\frac{1}{n\pi} \frac{1-3\nu^I}{1-\nu^I} J_o\left(in\pi \frac{a_o}{\ell_o}\right) - \right.$$

$$\left. \frac{a_o}{\ell_o} \frac{1-2\nu^I}{1-\nu^I}\, i\, J_1\left(in\pi \frac{a_o}{\ell_o}\right)\right\} -$$

$$i\,A_3^{II} \frac{E^{II}}{\left(1+\nu^{II}\right)\left(1-2\nu^{II}\right)} \frac{1}{1-\beta^{II}}\left\{\frac{1-2\nu^{II}}{1-\nu^{II}} J_o\left(in\pi \frac{a_o}{\ell_o}\right) + \right.$$

$$\left. \frac{\ell_o}{a_o} \frac{1-2\nu^{II}}{n\pi}\, i\, J_1\left(in\pi \frac{a_o}{\ell_o}\right)\right\} -$$

$$A_3^{II}\, a_2^{II} \frac{E^{II}}{\left(1+\nu^{II}\right)\left(1-2\nu^{II}\right)} \frac{1}{1-\beta^{II}}\left\{\frac{1-2\nu^{II}}{1-\nu^{II}}\, i\, H_o\left(in\pi \frac{a_o}{\ell_o}\right) - \right.$$

$$\left. \frac{\ell_o}{a_o} \frac{1-2\nu^{II}}{n\pi} H_1^{(1)}\left(in\pi \frac{a_o}{\ell_o}\right)\right\} -$$

$$- A_3^{II} a_3^{II} \frac{E^{II} \ell_o}{\left(1 + \nu^{II}\right)\left(1 - 2\nu^{II}\right)} \left\{\frac{1}{n\pi} \frac{1 - \nu^{II} - 2\left(\nu^{II}\right)^2}{1 - \nu^{II}} J_o\left(in\pi \frac{a_o}{\ell_o}\right) - \right.$$

$$\left. \frac{a_o}{\ell_o} \frac{1 - 2\nu^{II}}{1 - \nu^{II}} i J_1\left(in\pi \frac{a_o}{\ell_o}\right)\right\} +$$

$$i A_3^{II} a_4^{II} \frac{E^{II} \ell_o}{\left(1 + \nu^{II}\right)\left(1 - 2\nu^{II}\right)} \left\{\frac{1}{n\pi} \frac{1 - \nu^{II} - 2\left(\nu^{II}\right)^2}{1 - \nu^{II}} iH_o\left(in\pi \frac{a_o}{\ell_o}\right) + \right.$$

$$\left. \frac{a_o}{\ell_o} \frac{1 - 2\nu^{II}}{1 - \nu^{II}} H_1^{(1)}\left(in\pi \frac{a_o}{\ell_o}\right)\right\} = 0 \quad (168)$$

To simplify, the six constants resulting from equations (163) through (168) can be defined by

$$- i A_3^I = C_1$$

$$a_3^I A_3^I = C_2$$

$$- i A_3^{II} = C_3$$

$$a_2^{II} A_3^{II} = C_4$$

$$a_3^{II} A_3^{II} = C_5$$

$$- i a_4^{II} A_3^{II} = C_6 \quad (169)$$

The coefficients C_1 to C_6 are real numbers. Consequently, the solutions expressed in equations (145) through (150) can be written by additionally taking into account for ξ_3^I and ξ_3^{II} equations (153) through (158) in the following form:

Reinforcement Distortion in the Radial Direction:

$$\xi_1^I(r,z) = \sum_{n=1}^{\infty} \left\{ C_1 \, i \, J_1\left(i \frac{n\pi}{1-\beta^I} \frac{r}{\ell_o}\right) + C_2 \, r \, J_o\left(i \frac{n\pi}{1-\beta^I} \frac{r}{\ell_o}\right)\right\} \cos \frac{n\pi z}{\ell_o\left(1-\beta^I\right)} \tag{170}$$

Matrix Distortion in the Radial Direction:

$$\xi_1^{II}(r,z) = \sum_{n=1}^{\infty} \left\{ C_3 \, i \, J_1\left(i \frac{n\pi}{1-\beta^{II}} \frac{r}{\ell_o}\right) + C_4 \, H_1^{(1)}\left(i \frac{n\pi}{1-\beta^{II}} \frac{r}{\ell_o}\right) + C_5 \, r \, J_o\left(i \frac{n\pi}{1-\beta^{II}} \frac{r}{\ell_o}\right) + C_6 \, r \, i \, H_o^{(1)}\left(i \frac{n\pi}{1-\beta^{II}} \frac{r}{\ell_o}\right)\right\} \cos \frac{n\pi z}{\ell_o\left(1-\beta^{II}\right)} \tag{171}$$

Reinforcement Distortion in the Tangential Direction:

$$\xi_2^I(r,z) = 0 \tag{172}$$

Matrix Distortion in the Tangential Direction:

$$\xi_2^{II}\left(r,z\right) = 0 \tag{173}$$

Reinforcement Distortion in the Axial Direction:

$$\xi_3^{I}\left(r,z\right) = \frac{\nu^I}{1-\nu^I}\sum_{n=1}^{\infty}\left\{\left(C_1 - C_2\,\frac{2\,\ell_o\left(1-\beta^I\right)}{\pi n}\right) J_o\left(i\,\frac{n\pi}{1-\beta^I}\,\frac{r}{\ell_o}\right) + C_2\, i\, J_1\left(i\,\frac{n\pi}{1-\beta^I}\,\frac{r}{\ell_o}\right)\right\}\sin\frac{n\pi z}{\ell_o\left(1-\beta^I\right)} \tag{174}$$

Matrix Distortion in the Axial Direction:

$$\begin{aligned}\xi_3^{II} = \frac{\nu^{II}}{1-\nu^{II}}\sum_{n=1}^{\infty}\Bigg\{&\left(C_3 - C_5\,\frac{2\,\ell_o\left(1-\beta^{II}\right)}{\pi n}\right) J_o\left(i\,\frac{n\pi}{1-\beta^{II}}\,\frac{r}{\ell_o}\right) - \\ &\left(C_4 + C_6\,\frac{2\,\ell_o\left(1-\beta^{II}\right)}{\pi n}\right) i\, H_o^{(1)}\left(i\,\frac{n\pi}{1-\beta^{II}}\,\frac{r}{\ell_o}\right) + \\ &C_5\, r\, i\, J_1\left(i\,\frac{n\pi}{1-\beta^{II}}\,\frac{r}{\ell_o}\right) - \\ &C_6\, r\, H_1^{(1)}\left(i\,\frac{n\pi}{1-\beta^{II}}\,\frac{r}{\ell_o}\right)\Bigg\}\sin\frac{n\pi z}{\ell_o\left(1-\beta^{II}\right)}\end{aligned} \tag{175}$$

The Bessel and Hankel functions

$$J_o(ix)\ ,\ i\ J_1(ix)\ ,\ i\ H_o^{(1)}(ix)\ \text{and}\ \ H_1^{(1)}(ix)$$

must be recognized as real functions for $x > 0$. These functions are tabulated in the literature. The following identities interrelate the above functions and the so-called modified Bessel functions: (Ref. 3)

$$e^{-\frac{\alpha\pi i}{2}} J_\alpha(ix) = I_\alpha(x)$$

$$i\ e^{+\frac{\alpha\pi i}{2}} H_\alpha^{(1)}(ix) = \frac{2}{\pi} K_\alpha(x) \tag{176}$$

Especially

$$J_o(ix) = I_o(x)$$

$$e^{-\frac{\pi i}{2}} J_1(ix) = -i\ J_1(ix) = I_1(x)$$

$$i\ H_o^{(1)}(ix) = \frac{2}{\pi} K_o(x)$$

$$i^2\ H_1^{(1)}(ix) = -H_1^{(1)}(ix) = \frac{2}{\pi} K_1(x) \tag{177}$$

Introducing further abbreviations,

$$\frac{n\pi\ a_o}{\ell_o} = k_1 \tag{178}$$

$$\frac{n \pi b_o}{\ell_o} = k_2 \qquad (179)$$

$$\frac{E^I}{\left(1 + \nu^I\right)\left(1 - 2\nu^I\right)} = m_1 \qquad (180)$$

$$\frac{E^{II}}{\left(1 + \nu^{II}\right)\left(1 - 2\nu^{II}\right)} = m_2 \qquad (181)$$

In addition, the following is set:

$$\beta^I = \beta_1 \quad \beta^{II} = \beta_2 \quad \nu^I = \nu_1 \quad \nu^{II} = \nu_2$$

The following equation exists for the determination of the C_i ($i = 1$ to 6)

$$\sum_{i=1}^{6} B_{ji} C_i = D_j \quad \text{where } j = (1 \text{ to } 6)$$

The 36 B's are defined according to the equations used to this point of development:

$$B_{11} = -\frac{1}{k_1} I_1\left(k_1\right) \qquad (182)$$

$$B_{12} = \frac{\ell_o}{n\pi}\left(1 - \beta_1\right) I_o\left(k_1\right) \qquad (183)$$

$$B_{13} = \frac{1}{k_1} I_1\left(k_1\right) \tag{184}$$

$$B_{14} = \frac{1}{k_1}\frac{2}{\pi} K_1\left(k_1\right) \tag{185}$$

$$B_{15} = -\frac{\ell_o}{n\pi}\left(1 - \beta_2\right) I_o\left(k_1\right) \tag{186}$$

$$B_{16} = -\frac{\ell_o}{n\pi}\left(1 - \beta_2\right)\frac{2}{\pi} K_o\left(k_1\right) \tag{187}$$

$$B_{21} = -\frac{\nu_1}{1 - \nu_1} I_o\left(k_1\right) \tag{188}$$

$$B_{22} = \frac{\nu_1}{1 - \nu_1}\frac{\ell_o\left(1 - \beta_1\right)}{n\pi}\left[2 I_o\left(k_1\right) - k_1 I_1\left(k_1\right)\right] \tag{189}$$

$$B_{23} = -\frac{\nu_2}{1 - \nu_2} I_o\left(k_1\right) \tag{190}$$

$$B_{24} = \frac{\nu_2}{1 - \nu_2}\frac{2}{\pi} K_o\left(k_1\right) \tag{191}$$

$$B_{25} = \frac{\nu_2}{1 - \nu_2}\frac{\ell_o\left(1 - \beta_2\right)}{n\pi}\left[2 I_o\left(k_1\right) + k_1 I_1\left(k_1\right)\right] \tag{192}$$

$$B_{26} = \frac{\nu_2}{1 - \nu_2}\frac{\ell_o\left(1 - \beta_2\right)}{n\pi}\left[2\frac{2}{\pi} K_o\left(k_1\right) - k_1\frac{2}{\pi} K_1\left(k_1\right)\right] \tag{193}$$

$$B_{31} = 0 \tag{194}$$

$$B_{32} = 0 \tag{195}$$

$$B_{33} = \frac{1}{k_2}\frac{1 - 2\nu_2}{1 - \beta_2} I_1\left(k_2\right) + \frac{1 - 2\nu_2}{1 - \nu_2}\frac{1}{1 - \beta_2} I_o\left(k_2\right) \tag{196}$$

$$B_{34} = \frac{1}{k_2}\frac{1 - 2\nu_2}{1 - \beta_2}\frac{2}{\pi} K_1\left(k_2\right) + \frac{1 - 2\nu_2}{1 - \nu_2}\frac{1}{1 - \beta_2}\frac{2}{\pi} K_o\left(k_2\right) \tag{197}$$

$$B_{35} = -\frac{\ell_o}{n\pi}\left[\frac{2\nu_2^2 - \nu_2 + 1}{1 - \nu_2} I_o\left(k_2\right) - k_2\frac{1 - 2\nu_2}{1 - \nu_2} I_1\left(k_2\right)\right] \tag{198}$$

$$B_{36} = -\frac{\ell_o}{n\pi}\left[\frac{2\nu_2^2 - \nu_2 + 1}{1 - \nu_2}\frac{2}{\pi} K_o\left(k_2\right) + k_2\frac{1 - 2\nu_2}{1 - \nu_2}\frac{2}{\pi} K_1\left(k_2\right)\right] \tag{199}$$

$$B_{41} = 0 \tag{200}$$

$$B_{42} = 0 \tag{201}$$

$$B_{43} = -n\pi\, I_1\left(k_2\right) \tag{202}$$

$$B_{44} = -n\pi\frac{2}{\pi} K_1\left(k_2\right) \tag{203}$$

$$B_{45} = \ell_o\left(1 - \beta_2\right)\left[k_2\, I_o\left(k_2\right) + 2\nu_2\, I_1\left(k_2\right)\right] \tag{204}$$

$$B_{46} = \ell_o\left(1 - \beta_2\right)\left[k_2\frac{2}{\pi} K_o\left(k_2\right) - 2\nu_2\frac{2}{\pi} K_1\left(k_2\right)\right] \tag{205}$$

$$B_{51} = n\pi \frac{m_1\left(1 - 2\nu_1\right)}{1 - \nu_1} \frac{1}{1 - \beta_1} I_1\left(k_1\right) \tag{206}$$

$$B_{52} = - \frac{m_1\left(1 - 2\nu_1\right)}{\left(1 - \nu_1\right)} L_o \left[k_1 I_o\left(k_1\right) + 2\nu_1 I_1\left(k_1\right)\right] \tag{207}$$

$$B_{53} = - \frac{n\pi\, m_2 \left(1 - 2\nu_2\right)}{\left(1 - \nu_2\right)\left(1 - \beta_2\right)} I_1\left(k_1\right) - - - \tag{208}$$

$$B_{54} = - \frac{n\pi\, m_2\left(1 - 2\nu_2\right)}{\left(1 - \nu_2\right)\left(1 - \beta_2\right)} \frac{2}{\pi} K_1\left(k_1\right) - - - \tag{209}$$

$$B_{55} = m_2 \frac{\left(1 - 2\nu_2\right)}{1 - \nu_2} L_o \left[K_1 I_o\left(k_1\right) + 2\nu_2 I_1\left(k_1\right)\right] \tag{210}$$

$$B_{56} = \frac{m_2\left(1 - 2\nu_2\right)}{\left(1 - \nu_2\right)} L_o \left[K_1 \frac{2}{\pi} K_o\left(k_1\right) - 2\nu_2 \frac{2}{\pi} K_1\left(k_1\right)\right] \tag{211}$$

$$B_{61} = - \frac{m_1}{1 - \beta_1} \left[\frac{1 - 2\nu_1}{1 - \nu_1} I_o\left(k_1\right) - \frac{1 - 2\nu_1}{k_1} I_1\left(k_1\right)\right] \tag{212}$$

$$B_{62} = \frac{m_1 \ell_o}{n\pi} \left[\frac{1 - 3\nu_1}{1 - \nu_1} I_o\left(k_1\right) + \frac{1 - 2\nu_1}{1 - \nu_1} k_1 I_1\left(k_1\right)\right] \tag{213}$$

$$B_{63} = \frac{m_2}{1 - \beta_2} \left[\frac{1 - 2\nu_2}{1 - \nu_2} I_o\left(k_1\right) - \frac{1 - 2\nu_2}{k_1} I_1\left(k_1\right)\right] \tag{214}$$

$$B_{64} = \frac{m_2}{1 - \beta_2} \left[\frac{1 - 2\nu_2}{1 - \nu_2} \frac{2}{\pi} K_o\left(k_1\right) + \frac{1 - 2\nu_2}{k_1} \frac{2}{\pi} K_1\left(k_1\right)\right] \tag{215}$$

$$B_{65} = -\frac{m_2 \ell_o}{n\pi}\left[\frac{1 - \nu_2 - 2\nu_2^2}{1 - \nu_2} I_o\left(k_1\right) + \frac{1 - 2\nu_2}{1 - \nu_2} k_1 I_1\left(k_1\right)\right] \tag{216}$$

$$B_{66} = -\frac{m_2 \ell_o}{n\pi}\left[\frac{1 - \nu_2 - 2\nu_2^2}{1 - \nu_2} \frac{2}{\pi} K_o\left(k_1\right) - \frac{1 - 2\nu_2}{1 - \nu_2} k_1 \frac{2}{\pi} K_1\left(k_1\right)\right] \tag{217}$$

Further,

$$D_1 = -(-1)^n \frac{2 \ell_o}{n\pi}\left(\beta_1 - \beta_2\right) \tag{218}$$

$$D_2 = -(-1)^n \frac{2 \ell_o}{n\pi}\left(\beta_1 - \beta_2\right) \tag{219}$$

$$D_3 = 0 \tag{220}$$

$$D_4 = 0 \tag{221}$$

$$D_5 = 0 \tag{222}$$

$$D_6 = 0 \tag{223}$$

Numerical Determination of C's:

Given dimensions:

$a_o \quad b_o \quad L_o \quad E_1 \quad E_2 \quad \nu_1 \quad \nu_2$

$\beta_1 \quad \beta_2$

Basic computations:

$$k_1 = n\pi \frac{a_o}{L_o}$$

$$k_2 = \frac{n\pi\ b_o}{L_o}$$

$$m_1 = \frac{E_1}{\left(1 + \nu_1\right)\left(1 - 2\nu_1\right)}$$

$$\alpha_1 = \frac{m_1 \left(1 - 2\nu_1\right)}{\left(1 - \nu_2\right)\left(1 - \beta_1\right)}$$

$$\alpha_2 = \frac{m_2 \left(1 - 2\nu_2\right)}{\left(1 - \nu_2\right)\left(1 - \beta_2\right)}$$

Definitions:

$$F_1 = \frac{2\ I_o\left(k_1\right)\ I_1\left(k_1\right) - k_1\left[I_1^2\left(k_1\right) + I_o^2\left(k_1\right)\right]}{I_1\left(k_1\right)} \tag{224}$$

$$\gamma_1 = 2\nu_2\ \alpha_2\ I_1\left(k_1\right) - k_1\ I_o\left(k_1\right)\left[\alpha_1 - \alpha_2\right] \tag{225}$$

$$\gamma_2 = 2\nu_2\ \alpha_2\ K_1\left(k_1\right) + k_1\ K_o\left(k_1\right)\left[\alpha_1 - \alpha_2\right] \tag{226}$$

$$\gamma_3 = I_1\left(k_1\right)\left[2\left(1 - 3\nu_1 + \nu_1^2\right)\right] + k_1\left(1 - 2\nu_1\right)\left[\frac{I_1^2\left(k_1\right) - I_o^2\left(k_1\right)}{I_1\left(k_1\right)}\right] \tag{227}$$

$$\gamma_4 = \left(\alpha_2 - \alpha_1\right) I_o\left(k_1\right) + \frac{I_1\left(k_1\right)}{k_1}\left[\alpha_1\left(1 - \nu_1\right) - \alpha_2\left(1 - \nu_2\right)\right] \tag{228}$$

$$\gamma_5 = \alpha_2 K_o\left(k_1\right) + \alpha_1 \frac{K_1\left(k_1\right) I_o\left(k_1\right)}{I_1\left(k_1\right)} + \frac{K_1\left(k_1\right)}{k_1}\left[\alpha_2\left(1 - \nu_2\right) - \alpha_1\left(1 - \nu_1\right)\right] \tag{229}$$

$$\gamma_6 = \left\{k_1\left[\frac{\alpha_1 I_o^2\left(k_1\right)}{I_1\left(k_1\right)} - \alpha_2 I_1\left(k_1\right)\right] - I_o\left(k_1\right)\left[\alpha_1\left(1 - \nu_1\right) + \alpha_2\left(1 + \nu_2\right)\right]\right\} \times \frac{L_o}{n\pi} \tag{230}$$

$$\gamma_7 = \alpha_2 k_1 K_1\left(k_1\right) + \frac{\alpha_1 k_1 I_o\left(k_1\right) K_o\left(k_1\right)}{I_1\left(k_1\right)} - \left[\alpha_2\left(1 + \nu_2\right) + \alpha_1\left(1 - \nu_1\right)\right] K_o\left(k_1\right) \tag{231}$$

$$\gamma_8 = \left(1 - \nu_1\right) - \frac{k_1 I_o\left(k_1\right)}{I_1\left(k_1\right)} \tag{232}$$

$$F_2 = \frac{\nu_2\left(1 - \nu_1\right) K_o\left(k_1\right) I_1\left(k_1\right) - \nu_1\left(1 - \nu_2\right) K_1\left(k_1\right) I_o\left(k_1\right)}{I_1\left(k_1\right)} \tag{233}$$

$$F_3 = \nu_2\left(1 - \nu_1\right)\left[2 I_o\left(k_1\right) + k_1 I_1\left(k_1\right)\right] + \nu_1\left(1 - \nu_2\right) k_1 \frac{I_o^2\left(k_1\right)}{I_1\left(k_1\right)} \tag{234}$$

$$F_4 = \nu_2\left(1 - \nu_1\right)\left[2\,K_o\left(k_1\right) - k_1\,K_1\left(k_1\right)\right] + \frac{\nu_1\left(1 - \nu_2\right)\,k_1\,K_o\left(k_1\right)\,I_o\left(k_1\right)}{I_1\left(k_1\right)} \qquad (235)$$

$$F_5 = \left(1 - \nu_1\right) - \frac{\nu_1\,k_1\,I_o\left(k_1\right)}{I_1\left(k_1\right)} \qquad (236)$$

$$F_6 = \left(1 - \nu_2\right)\,I_1\left(k_2\right) + k_2\,I_o\left(k_2\right) \qquad (237)$$

$$F_7 = \left(1 - \nu_2\right)\,K_1\left(k_2\right) + k_2\,K_o\left(k_2\right) \qquad (238)$$

$$F_8 = \left(2\nu_2^2 - \nu_2 + 1\right)\,I_o\left(k_2\right) - k_2\left(1 - 2\nu_2\right)\,I_1\left(k_2\right) \qquad (239)$$

$$F_9 = \left(2\nu_2^2 - \nu_2 + 1\right)\,K_o\left(k_2\right) + k_2\left(1 - 2\nu_2\right)\,K_1\left(k_2\right) \qquad (240)$$

$$F_{10} = k_2\,I_o\,k_2 + 2\nu_2\,I_1\,k_2 \qquad (241)$$

$$F_{11} = k_2\,K_o\left(k_2\right) - 2\nu_2\,K_1\left(k_2\right) \qquad (242)$$

$$F_{12} = \alpha_1 - \alpha_2 - \frac{2\,\alpha_1\left[\nu_2\left(1 - \nu_1\right) + \nu_1\left(1 - \nu_2\right)\right]}{\left(1 - \nu_2\right)}\,\frac{I_o\left(k_1\right)}{F_1\left(k_1\right)} \qquad (243)$$

$$F_{13} = 2n\,K_1\left(k_1\right)\left[\alpha_1 - \alpha_2\right] + \frac{4\,\alpha_1\,n}{1 - \nu_2}\,\frac{I_1\left(k_1\right)\,F_2\left(k_1\right)}{F_1\left(k_1\right)} \qquad (244)$$

$$F_{14} = \gamma_1 + \frac{2\,\alpha_1}{1 - \nu_2}\,\frac{I_1\left(k_1\right)\,F_3\left(k_1\right)}{F_1\left(k_1\right)} \tag{245}$$

$$F_{15} = \frac{2\,\alpha_1}{1 - \nu_2}\,\frac{I_1\left(k_1\right)\,F_4\left(k_1\right)}{F_1\left(k_1\right)} - \gamma_2 \tag{246}$$

$$F_{16} = k_1 + \frac{2\,I_1\left(k_1\right)\,F_5\left(k_1\right)}{F_1\left(k_1\right)} \tag{247}$$

$$F_{17} = \gamma_4 + \frac{m_1\left[\nu_2\left(1 - \nu_1\right) + \nu_1\left(1 - \nu_2\right)\right]}{\left(1 - \nu_1\right)\left(1 - \nu_2\right)\,\nu_1\left(1 - \beta_1\right)}\,\frac{\gamma_3\,I_0\left(k_1\right)}{F_1\left(k_1\right)} \tag{248}$$

$$F_{18} = \gamma_5 + \frac{\alpha_1}{\left(1 - 2\nu_1\right)\,\nu_1\left(1 - \nu_2\right)}\,\frac{F_2\left(k_1\right)\,\alpha_3}{F_1\left(k_1\right)} \tag{249}$$

$$F_{19} = \gamma_6 - \frac{\alpha_1\,L_0}{\left(1 - 2\nu_1\right)\,\nu_1\,m\pi\left(1 - \nu_2\right)}\left[\frac{F_3\left(k_1\right)}{F_1\left(k_1\right)}\,\gamma_3\right] \tag{250}$$

$$F_{20} = \gamma_7 - \frac{\alpha_1\,F_4\,\gamma_3}{\left(1 - 2\nu_1\right)\,\nu_1\left(1 - \nu_2\right)\,F_1\left(k_1\right)} \tag{251}$$

$$F_{21} = \gamma_8 - \frac{1}{\left(1 - 2\nu_1\right)\nu_1}\,\frac{F_3\,\gamma_3}{F_1} \tag{252}$$

$$F_{22} = \frac{I_1\left(k_2\right)\,F_7}{F_6} - K_1\left(k_2\right) \tag{253}$$

$$F_{23} = F_{10} - \frac{k_2}{1 - 2\nu_2}\,\frac{F_8}{F_6}\,I_1\left(k_2\right) \tag{254}$$

$$F_{24} = F_{11} - \frac{k_2}{1 - 2\nu_2} \frac{F_9}{F_6} I_1(k_2) \tag{255}$$

$$F_{25} = F_{13} - \frac{F_7 F_{12} I_1(k_1)}{F_6} \tag{256}$$

$$F_{26} = F_{14} + \frac{k_2}{1 - 2\nu_2} \frac{F_8 F_{12} I_1(k_1)}{F_6} \tag{257}$$

$$F_{27} = F_{15} + \frac{k_2}{1 - 2\nu_2} \frac{F_9 F_{12} I_1(k_1)}{F_6} \tag{258}$$

$$F_{28} = F_{18} + \frac{F_7 F_{17}}{F_6} \tag{259}$$

$$F_{29} = F_{19} + \frac{k_2 L_o}{n\pi(1 - 2\nu_2)} \frac{F_8 F_{17}}{F_6} \tag{260}$$

$$F_{30} = F_{20} + \frac{k_2}{\pi(1 - 2\nu_2)} \frac{F_9}{F_6} F_{17} \tag{261}$$

$$F_{31} = F_{26} - \frac{1}{2n} \frac{F_{23} F_{25}}{F_{22}} \tag{262}$$

$$F_{32} = 2 F_{27} - \frac{1}{n} \frac{F_{24} F_{25}}{F_{22}} \tag{263}$$

$$F_{33} = F_{29} + \frac{L_o(1 - \beta_2)}{n\pi} \frac{F_{23}}{F_{22}} F_{28} \tag{264}$$

$$F_{34} = F_{30} + \frac{1}{\pi}\frac{F_{24}}{F_{22}} F_{28} \tag{265}$$

$$F_{35} = \frac{2\,L_o}{n} F_{34} - \frac{F_{32}\,F_{33}}{F_{31}} \tag{266}$$

$$F_{36} = F_{21} - \frac{n\pi}{L_o}\frac{F_{16}}{F_{31}} F_{33} \tag{267}$$

Then

$$C_6 = \frac{\pi\,\alpha_1\,N_1}{1 - \beta_2}\frac{F_{36}}{F_{35}} \tag{268}$$

$$C_5 = \frac{n\pi\,\alpha_1\,N_1}{L_o\left(1 - \beta_2\right)}\frac{F_{16}}{F_{31}} - \frac{1}{\pi}\frac{F_{32}}{F_{31}} C_6 \tag{269}$$

$$\begin{aligned} C_4 &= -\left[\frac{L_o\left(1 - \beta_2\right)}{n\pi}\frac{F_{24}}{F_{22}} C_6 + \frac{L_o\left(1 - \beta_2\right)}{2n}\frac{F_{23}}{F_{22}} C_5\right] \\ &= -\frac{L_o\left(1 - \beta_2\right)}{n}\left[\frac{F_{24}\,C_6}{\pi\,F_{22}} + \frac{F_{23}\,C_5}{2\,F_{22}}\right] \\ &= \left(1 - \beta_2\right)\frac{2}{\pi}\left\{\frac{k_2\,L_o}{n\left(1 - 2\nu_2\right)}\left[\frac{C_6}{\pi}\frac{F_9}{F_6} + \frac{C_5}{2}\frac{F_8}{F_6}\right] - \frac{C_4}{1 - \beta_2}\frac{F_7}{F_6}\right\} \end{aligned} \tag{270}$$

$$C_2 = \frac{1}{F_1\,\nu_1\left(1 - \beta_1\right)\left(1 - \nu_2\right)}\left\{\frac{n\pi\,N_1\left(1 - \nu_2\right)}{L_o} F_5 - \right.$$

$$- \frac{2\left(1 - \beta_2\right)}{\pi} F_4 C_6 - \frac{\left(1 - \beta_2\right)}{1} F_3 C_5 - \frac{2n}{L_o} F_2 C_4 +$$

$$\left. \frac{\left[\nu_2\left(1 - \nu_1\right) + \nu_1\left(1 - \nu_2\right)\right]}{L_o} n\pi\, I_o\left(k_1\right) C_3 \right\} \qquad (271)$$

$$C_1 = \frac{2}{\pi} \frac{K_1\left(k_1\right)}{I_1\left(k_1\right)} C_4 + C_3 - \frac{k_1}{I_1\left(k_1\right)} \left\{ N_1 + \right.$$

$$\frac{L_o}{n\pi} \left[\left(1 - \beta_2\right) \frac{2}{\pi} K_o\left(k_1\right) C_6 + \left(1 - \beta_2\right) I_o\left(k_1\right) C_5 - \right.$$

$$\left. \left. \left(1 - \beta_1\right) I_o\left(k_1\right) C_2 \right] \right\} \qquad (272)$$

THE INFINITE FIBER IN A RESIN CYLINDER

The basic equations (1) through (10) are, in this case, also valid, while the boundary conditions change insofar as there is no finite length expression given. After separation of the variables by introducing a Bernoulli product of the form

$$\xi = R(r)\, e^{\pm i\lambda z}$$

and further introducing in the obtained differential equation the potential

$$\phi(x) = \frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \left(1 - \frac{1}{x^2}\right) R$$

where $x = \lambda r$ and λ is the eigenvalue. The following solutions are obtained

$$\zeta_1^{I}\left(r,z\right) = A_1 r + \left[B_1 r\, J_o\left(i\lambda r\right) + B_2 i\, J_1\left(i\lambda r\right)\right] \cdot \cos \lambda z \tag{273}$$

$$\zeta_1^{II}\left(r,z\right) = A_2 r + \frac{A_3}{r} + \left[B_3 ir\, H_o^{(1)}\left(i\lambda r\right) + B_4\, H_1^{(1)}\left(i\lambda r\right)\right] \cos \lambda z \tag{274}$$

$$\zeta_3^{I}\left(r,z\right) = A_4 + \left[B_5\, J_o\left(i\lambda r\right) + B_6 ir\, J_1\left(i\lambda r\right)\right] \cos \lambda z \tag{275}$$

$$\zeta_3^{II}\left(r,z\right) = A_5 + \left[B_7 i\, H_o^{(1)}\left(i\lambda r\right) + B_8 r\, H_1^{(1)}\left(i\lambda r\right)\right] \cos \lambda z \tag{276}$$

Applying the solutions to the corresponding boundary conditions, eight equations for the integration constants B's result. The system determinant was homogeneous and the following four eigenvalue equations were obtained:

$$J_o^2\left(\lambda b\right) + J_1^2\left(\lambda b\right) = 0 \tag{277}$$

$$H_1^{(1)^2}\left(\lambda b\right) + H_o^{(1)^2}\left(\lambda b\right) = 0 \tag{278}$$

$$J_o^2\left(\lambda a\right) + J_1^2\left(\lambda a\right) - \frac{2}{\lambda a} J_o\left(\lambda a\right) J_1\left(\lambda a\right) = 0 \tag{279}$$

$$H_o^{(1)^2}\left(\lambda b\right) + H_1^{(1)^2}\left(\lambda b\right) - \frac{2}{\lambda b} H_o^{(1)}\left(\lambda b\right) H_1^{(1)}\left(\lambda b\right) = 0 \tag{280}$$

The methods given in Computation of Hankel Functions* did not result in conjugated complex roots. This indicates that all B's are zero in the solutions expressed in equations (273) through (276), and the stresses for the infinite fiber become

$$\sigma_{11}^{I}(r,z) = - \frac{E^{I}}{\left(1+\nu^{I}\right)\left(1-2\nu^{I}\right)} \frac{\beta^{I} - \beta^{II} + \left(\alpha^{I} - \alpha^{II}\right) T_o}{1 + \frac{E^{I}}{E^{II}} \frac{1+\nu^{II}}{\left(1+\nu^{I}\right)\left(1-2\nu^{I}\right)} \frac{1 + \frac{a^2}{b^2}\left(1-2\nu^{II}\right)}{1 - \frac{a^2}{b^2}}} \tag{281}$$

$$\sigma_{11}^{II}(r,z) = + \frac{E^{II}}{1+\nu^{II}} \frac{\left[\beta^{I} - \beta^{II} + \left(\alpha^{I} - \alpha^{II}\right) T_o\right]\left[\left(\frac{a}{b}\right)^2 - \frac{a^2}{r^2}\right]}{1 + \frac{a^2}{b^2}\left(1-2\nu^{II}\right) + \frac{E^{II}}{E^{I}} \frac{\left[1+\nu^{I}\right]\left[1-2\nu^{I}\right]}{1+\nu^{II}} \left(1 - \frac{a^2}{b^2}\right)} \tag{282}$$

$$\sigma_{22}^{I}(r,z) = \sigma_{11}^{I}(r,z) \tag{283}$$

$$\sigma_{22}^{II}(r,z) = + \frac{E^{II}}{\left(1+\nu^{II}\right)\left(1-2\nu^{II}\right)} \frac{\left[\beta^{I} - \beta^{II} + \left(\alpha^{I} - \alpha^{II}\right) T_o\right]\left[\frac{a^2}{b^2} - \frac{a^2}{r^2}\right]}{1 + \frac{a^2}{b^2}\left(1-2\nu^{II}\right) + \frac{E^{II}}{E^{I}} \frac{\left[1+\nu^{I}\right]\left[1-2\nu^{I}\right]}{1+\nu^{II}} \left(1 - \frac{a^2}{b^2}\right)} \tag{284}$$

$$\sigma_{33}^{I}(r,z) = - \frac{2\nu^{I} E^{I}}{\left[1+\nu^{I}\right]\left[1-2\nu^{I}\right]} \frac{\beta^{I} - \beta^{II} + \left(\alpha^{I} - \alpha^{II}\right) T_o}{1 + \frac{E^{I}}{E^{II}} \frac{1+\nu^{II}}{\left(1+\nu^{I}\right)\left(1-2\nu^{I}\right)} \frac{1 + \frac{a^2}{b^2}\left(1-2\nu^{II}\right)}{1 - \frac{a^2}{b^2}}} \tag{285}$$

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* Computation of Hankel Functions, National Bureau of Standards Report No. 216.

$$= 2\nu^{I}\,\sigma_{11}^{I}\left(r,z\right) \tag{286}$$

$$\sigma_{33}^{II}\left(r,z\right) = \frac{2\,E^{II}\,\nu^{II}}{\left(1+\nu^{I}\right)\left(1-2\nu^{I}\right)}\;\frac{\left[\beta^{I} - \beta^{II} + \left(\alpha^{I} - \alpha^{II}\right)T_{o}\right]\frac{a^{2}}{b^{2}}\left(1-2\nu^{II}\right)}{1 + \frac{a^{2}}{b^{2}}\left(1-2\nu^{II}\right) + \frac{E^{II}}{E^{I}}\,\frac{\left(1+\nu^{I}\right)\left(1-2\nu^{I}\right)}{1+\nu^{II}}\left(1 - \frac{a^{2}}{b^{2}}\right)} \tag{287}$$

$$\sigma_{12}^{I} = \sigma_{12}^{II} = \sigma_{13}^{I} = \sigma_{13}^{II} = \sigma_{23}^{I} = \sigma_{23}^{II} = 0 \tag{288}$$

Since the infinite undistrubed fiber is not realizable, equations (280) through (288) are values which are difficult to verify by test; however, they can be used to provide an estimate of the magnitude of residual stresses.

For instance, if the relation between the reinforcement and resin modulus were

$$E^{I} \approx 20\,E^{II}$$

$$\nu_{1} = \nu_{2} = 1/4 \quad \text{and} \quad b \gg a$$

and assuming that the total difference of contraction including polymerization were 4%, then by using equation (281) the following would exist:

$$\sigma_{11}^{I}\left(a,z\right) \approx \frac{E^{I}}{70.0}$$

This indicates that the radial compression of a glass fiber with a modulus of 11 x 10^{6} psi would be 15,700 psi on the surface of the fiber ($r = a$) .

For $b = 10\,a$, the axial, tensile stress in the resin would be 14,000 psi which would cause cracking; the resin would therefore become a finite length and the equation for infinite length would not be adequate. For this reason, no more emphasis is given to the infinite length fiber with assumed boundary conditions.

The problem would be different if the boundary conditions that are not physically imposed were derived in the form of a differential equation which provides for cracking in certain unknown distances.

THE MULTIFIBER PROBLEM

Assume that an infinite number of fibers are packed so that in the limiting case, the tightest packing is possible. A configuration such as this would possess a certain symmetry which would allow for rigorous analytical treatment.

Using the configuration depicted in Figure 3, any one fiber can be considered as the central fiber surrounded by six others (all fibers are imbedded in the matrix).

Boundary Conditions for the Multifiber Problem

By assuming three-dimensional shrinkage, a boundary condition is obtained on the interface between fiber and resin which is similar to the boundary condition for a single fiber in a resin cylinder, the only difference being that the distortions will depend also on the angle φ

$$\xi_1^{I}\left[a_o\left(1-\beta_1\right),\ \varphi\ ,\ z\left(1-\beta_1\right)\right]-\xi_1^{II}\left[a_o\left(1-\beta_2\right),\ \varphi\ ,\ z\left(1-\beta_2\right)\right]$$

$$= a_o\left(\beta_1-\beta_2\right) \qquad (289)$$

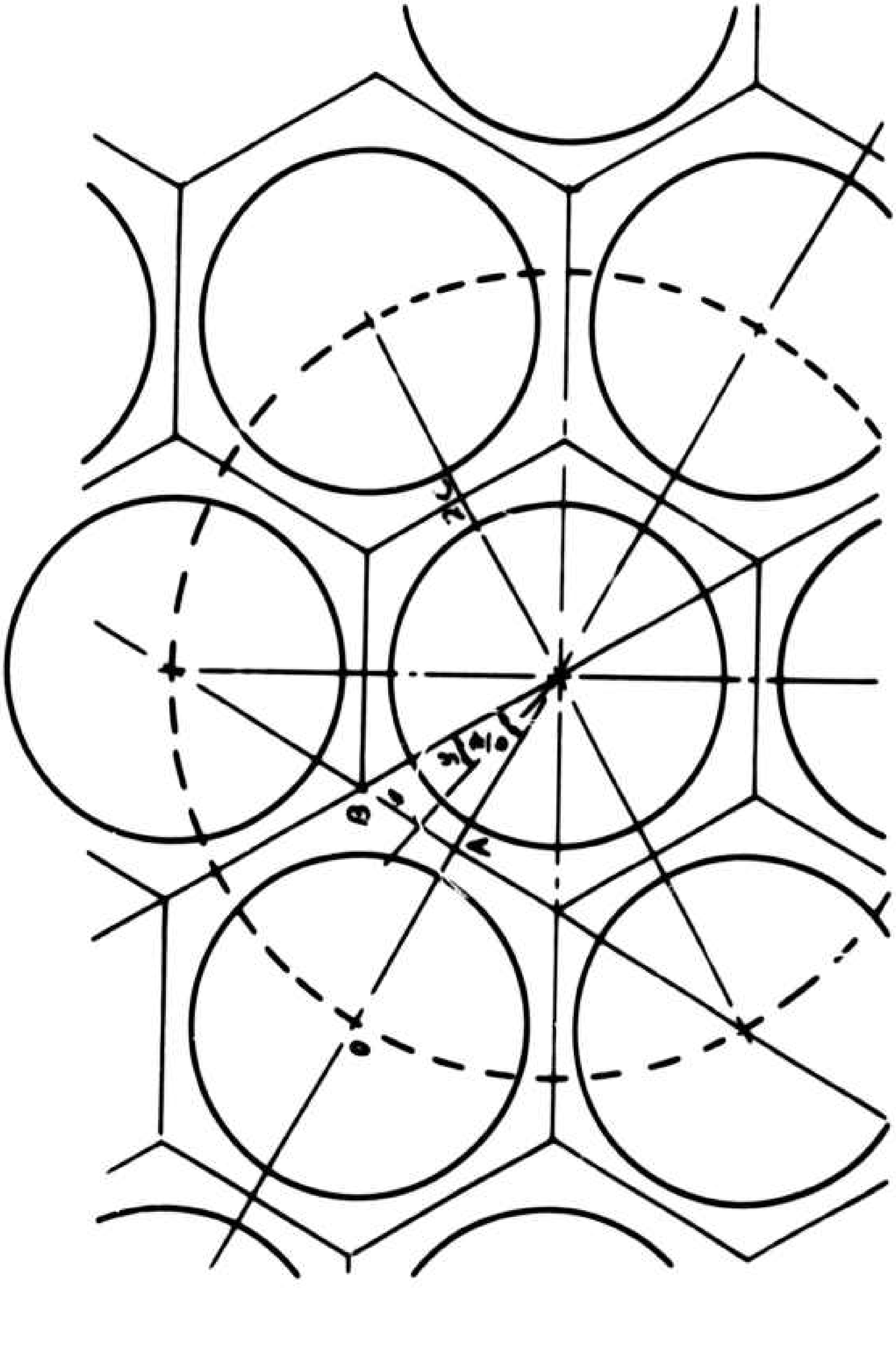

Figure 3.

$$\xi_2^I\left[a_o\left(1-\beta_1\right),\ \varphi,\ z\left(1-\beta_1\right)\right]-\xi_2^{II}\left[a_o\left(1-\beta_2\right),\ \varphi,\ z\left(1-\beta_2\right)\right]=0 \tag{290}$$

$$\xi_3^I\left[a_o\left(1-\beta_1\right),\ \varphi,\ z\left(1-\beta_1\right)\right]-\xi_3^{II}\left[a_o\left(1-\beta_2\right),\ \varphi,\ z\left(1-\beta_2\right)\right]$$

$$=z\left(\beta_1-\beta_2\right) \tag{291}$$

Where ξ_1^I is the displacement in direction 1 (or r) in the reinforcement, a_o is the radius of the fiber before distortion through shrinkages takes place, and z is, similarly, the coordinate before distortion.

Referring to Figure 3, it can be assumed for reasons of symmetry that the hexagonals will remain regular during the shrinking process, and will become proportionately smaller as the total composite shrinks.

The displacement vector perpendicular to OB must vanish if the triangle sites OAB remain straight.

For the reinforcement:

$$\xi_2^I\left[r\left(1-\beta^I\right),\ 0,\ z\left(1-\beta^I\right)\right]=0 \tag{292}$$

For the matrix:

$$\xi_2^{II}\left[r\left(1-\beta^I\right),\ 0,\ z\left(1-\beta^{II}\right)\right]=0 \tag{293}$$

For the OA line:

In the reinforcement

$$\xi_2^I\left[r\left(1-\beta^I\right),\ \frac{\pi}{6},\ z\left(1-\beta_1\right)\right]=0 \tag{294}$$

and in the matrix:

$$\xi_2^{II}\left[r\left(1 - \beta^{II}\right), \frac{\Pi}{6}, z\left(1 - \beta^{II}\right)\right] = 0 \qquad (295)$$

Along the AB line:

$$r = \frac{c_o}{\cos\left(\frac{\Pi}{6} - \varphi\right)}$$

and

$$\xi_1^{II}\left[\frac{c_o\left(1 - \beta^{II}\right)}{\cos\left(\frac{\Pi}{6} - \varphi\right)}, \varphi, z\left(1 - \beta^{II}\right)\right]\cos\left(\frac{\Pi}{6} - \varphi\right) -$$

$$\xi_2^{II}\left[\frac{c_o\left(1 - \beta^{II}\right)}{\cos\left(\frac{\Pi}{6} - \varphi\right)}, \varphi, z\left(1 - \beta^{II}\right)\right]\sin\left(\frac{\Pi}{6} - \varphi\right) = 0 \qquad (296)$$

Equations (292) through (295) must be valid for all values of r and z, and equation (296) for all values of φ in the region where

$$0 \leq \varphi \leq \frac{\Pi}{6}$$

Since the triangle OAB repeats 12 times in each hexagonal, it is possible to solve the problem by considering OAB only.

Differential Equations for Displacements

General partial differential equations for distortions considering the cylindrical coordinate system were derived.* These equations

$$\frac{1-\nu}{1-2\nu}\frac{\partial^2 \xi_1}{\partial r^2} + \frac{1}{2\,r^2}\frac{\partial^2 \xi_1}{\partial \varphi^2} + \frac{1}{2}\frac{\partial^2 \xi_1}{\partial z^2} + \frac{1}{2(1-2\nu)}\frac{1}{r}\frac{\partial^2 \xi_1}{\partial r \partial \varphi} +$$

$$\frac{1}{2(1-2\nu)}\frac{\partial^2 \xi_3}{\partial r \partial z} - \frac{3-4\nu}{2(1-2\nu)}\frac{1}{r^2}\frac{\partial \xi_2}{\partial \varphi} +$$

$$\frac{1-\nu}{1-2\nu}\frac{1}{r}\left(\frac{\partial \xi_1}{\partial r} - \frac{\xi_1}{r}\right) = 0 \qquad (297)$$

$$\frac{1}{2}\frac{\partial^2 \xi_2}{\partial r^2} + \frac{1-\nu}{1-2\nu}\frac{1}{r^2}\frac{\partial^2 \xi_2}{\partial \varphi^2} + \frac{1}{2}\frac{\partial^2 \xi_2}{\partial z^2} + \frac{1}{2(1-2\nu)}\frac{1}{r}\frac{\partial^2 \xi_1}{\partial r \partial \varphi} +$$

$$\frac{1}{2(1-2\nu)}\frac{1}{r}\frac{\partial^2 \xi_3}{\partial \varphi\,\partial z} + \frac{1}{2r}\left(\frac{3-4\nu}{1-2\nu}\frac{1}{r}\frac{\partial \xi_1}{\partial \varphi} + \frac{\partial \xi_2}{\partial r} - \frac{\xi_2}{r}\right) = 0 \qquad (298)$$

$$\frac{1}{2}\frac{\partial^2 \xi_3}{\partial r^2} + \frac{1}{2\,r^2}\frac{\partial^2 \xi_3}{\partial \varphi^2} + \frac{1-\nu}{1-2\nu}\frac{\partial^2 \xi_3}{\partial z^2} + \frac{1}{2(1-2\nu)}\frac{\partial^2 \xi_1}{\partial r \partial z} +$$

$$\frac{1}{2(1-2\nu)}\frac{1}{r}\frac{\partial^2 \xi_2}{\partial \varphi \partial z} + \frac{1}{2(1-2\nu)}\frac{1}{r}\frac{\partial \xi_1}{\partial z} + \frac{1}{2r}\frac{\partial \xi_3}{\partial r} = 0 \qquad (299)$$

* Equation (A73) in the Appendix.

The following identities are noted.

$$\frac{1}{2(1-2\nu)} = \frac{1-\nu}{1-2\nu} - \frac{1}{2} \qquad \frac{3-4\nu}{2(1-2\nu)} = \frac{1-\nu}{1-2\nu} + \frac{1}{2}$$

These are used in equations (297) through (299), which on rearrangement and simplification become the following sequential equations:

$$\frac{1-\nu}{1-2\nu} \frac{\partial}{\partial r}\left[\frac{1}{r} \frac{\partial}{\partial r}\left(r\, \xi_1\right) + \frac{\partial \xi_3}{\partial z} + \frac{1}{r} \frac{\partial \xi_2}{\partial \varphi}\right] +$$

$$\frac{1}{2r} \frac{\partial}{\partial z}\left[r\left(\frac{\partial \xi_1}{\partial z} - \frac{\partial \xi_3}{\partial r}\right)\right] + \frac{1}{2} \frac{\partial}{\partial \varphi}\left[\frac{1}{r^2} \frac{\partial \xi_1}{\partial \varphi} - \frac{1}{r^2} \xi_2 - \frac{1}{r} \frac{\partial \xi_2}{\partial r}\right] = 0 \qquad (300)$$

$$\frac{1-\nu}{1-2\nu} \frac{1}{r} \frac{\partial}{\partial \varphi}\left[\frac{1}{r} \frac{\partial \xi_2}{\partial \varphi} + \frac{\partial \xi_3}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r}\left(r\, \xi_1\right)\right] +$$

$$\frac{1}{2} \frac{\partial}{\partial r}\left\{\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\, \xi_2\right) - \frac{\partial \xi_1}{\partial \varphi}\right]\right\} + \frac{1}{2} \frac{\partial}{\partial z}\left[\frac{\partial \xi_2}{\partial z} - \frac{1}{r} \frac{\partial \xi_3}{\partial \varphi}\right] = 0 \qquad (301)$$

$$\frac{1-\nu}{1-2\nu} \frac{\partial}{\partial z}\left[\frac{1}{r} \frac{\partial}{\partial r}\left(r\, \xi_1\right) + \frac{\partial \xi_3}{\partial z} + \frac{1}{r} \frac{\partial \xi_2}{\partial \varphi}\right] +$$

$$\frac{1}{2r} \frac{\partial}{\partial r}\left[r\left(\frac{\partial \xi_3}{\partial r} - \frac{\partial \xi_1}{\partial z}\right)\right] + \frac{1}{2r} \frac{\partial}{\partial \varphi}\left[\frac{1}{r} \frac{\partial \xi_3}{\partial \varphi} - \frac{\partial \xi_2}{\partial z}\right] = 0 \qquad (302)$$

Some recurring functions in the previous equations are noted and defined as F's as follows.

$$\frac{1-\nu}{1-2\nu}\left(\frac{1}{r} \frac{\partial}{\partial r}\left(r\, \xi_1\right) + \frac{\partial \xi_3}{\partial z} + \frac{1}{r} \frac{\partial \xi_2}{\partial \varphi}\right) = F_1 \qquad (303)$$

$$\frac{r}{2}\left(\frac{\partial \xi_1}{\partial z} - \frac{\partial \xi_3}{\partial r}\right) = F_2 \tag{304}$$

$$\frac{1}{2r^2}\left[\frac{\partial \xi_1}{\partial \varphi} - \frac{\partial}{\partial r}\left(r\ \xi_2\right)\right] = F_3 \tag{305}$$

$$\frac{1}{2}\left[\frac{\partial \xi_2}{\partial z} - \frac{1}{r}\ \frac{\partial \xi_3}{\partial \varphi}\right] = F_4 \tag{306}$$

Using equations (303) through (306) in equations (300) through (302) yields the following partial differential equations:

$$\frac{\partial F_1}{\partial r} + \frac{1}{r}\ \frac{\partial F_2}{\partial z} + \frac{\partial F_3}{\partial \varphi} = 0 \tag{307}$$

$$\frac{1}{r}\ \frac{\partial F_1}{\partial \varphi} - \frac{\partial}{\partial r}\left(r\ F_3\right) + \frac{\partial F_4}{\partial z} = 0 \tag{308}$$

$$\frac{\partial F_1}{\partial z} - \frac{1}{r}\ \frac{\partial F_2}{\partial r} - \frac{1}{r}\ \frac{\partial F_4}{\partial \varphi} = 0 \tag{309}$$

Now F's can be considered as potential functions. First, efforts would be made to find them. In addition to the previous three equations, the procedure that follows renders one more equation, interrelating the F's.

Multiplying equation (305) by r^2 and equation (306) by r yields

$$\frac{1}{2}\left[\frac{\partial \xi_1}{\partial \varphi} - \frac{\partial}{\partial r}\left(r\ \xi_2\right)\right] = r^2\ F_3 \tag{310}$$

and

$$\frac{1}{2}\left[\frac{\partial\left(r\ \xi_2\right)}{\partial z} - \frac{\partial \xi_3}{\partial \varphi}\right] = r\ F_4 \tag{311}$$

Partial differentiation of equation (310) with respect to z and equation (311) with respect to r gives the following:

$$\frac{1}{2}\left[\frac{\partial^2 \xi_1}{\partial z \partial \varphi} - \frac{\partial^2\left(r\ \xi_2\right)}{\partial z \partial r}\right] = r^2\ \frac{\partial F_3}{\partial z} \tag{312}$$

$$\frac{1}{2}\left[\frac{\partial^2\left(r\ \xi_2\right)}{\partial r \partial z} - \frac{\partial^2 \xi_3}{\partial r \partial \varphi}\right] = \frac{\partial}{\partial r}\left(r\ F_4\right) \tag{313}$$

Summation of equations (312) and (313) gives

$$\frac{1}{2}\left[\frac{\partial^2 \xi_1}{\partial z \partial \varphi} - \frac{\partial^2 \xi_3}{\partial r \partial \varphi}\right] = r^2\ \frac{\partial F_3}{\partial z} + \frac{\partial}{\partial r}\left(r\ F_4\right) \tag{314}$$

Partial differentiation of equation (304) with respect to φ gives

$$\frac{r}{2}\left[\frac{\partial^2 \xi_1}{\partial \varphi \partial z} - \frac{\partial^2 \xi_3}{\partial \varphi \partial r}\right] = \frac{\partial F_2}{\partial \varphi}$$

i.e.,

$$\frac{1}{2}\left[\frac{\partial^2 \xi_1}{\partial z \partial \varphi} - \frac{\partial^2 \xi_3}{\partial r \partial \varphi}\right] = \frac{1}{r}\ \frac{\partial F_2}{\partial \varphi} \tag{315}$$

Noting that left-hand sides of equations (314) and (315) are identical, one obtains

$$\frac{1}{r}\ \frac{\partial F_2}{\partial \varphi} - r^2\ \frac{\partial F_3}{\partial z} - \frac{\partial\left(r\ F_4\right)}{\partial r} = 0 \tag{316}$$

Separation of the Potentials:

F_1:

Multiplying equation (308) by r and partially differentiating with respect to z, one obtains

$$r \frac{\partial^2 F_1}{\partial z^2} - \frac{\partial^2 F_2}{\partial z \partial r} - \frac{\partial^2 F_4}{\partial z \partial \varphi} = 0 \tag{317}$$

Partial differentiation of equation (309) with respect to φ gives

$$\frac{1}{r} \frac{\partial^2 F_1}{\partial \varphi^2} - \frac{\partial^2}{\partial \varphi \partial r} \left(r \, F_3 \right) + \frac{\partial^2 F_4}{\partial \varphi \partial z} = 0 \tag{318}$$

Summation of equations (317) and (318) renders

$$r \frac{\partial^2 F_1}{\partial z^2} + \frac{1}{r} \frac{\partial^2 F_1}{\partial \varphi^2} - \frac{\partial^2 F_2}{\partial z \partial r} - \frac{\partial^2}{\partial \varphi \partial z} \left(r \, F_3 \right) = 0$$

i.e.,

$$r \frac{\partial^2 F_1}{\partial z^2} + \frac{1}{r} \frac{\partial^2 F_1}{\partial \varphi^2} - \frac{\partial^2 F_2}{\partial z \partial r} - \frac{\partial}{\partial r} \left(r \, \frac{\partial F_3}{\partial \varphi} \right) = 0 \tag{319}$$

From equation (307),

$$\frac{\partial F_3}{\partial \varphi} = - \frac{\partial F_1}{\partial r} + \frac{1}{r} \frac{\partial F_2}{\partial z} \tag{320}$$

Substitution of $\frac{\partial F_3}{\partial \varphi}$ in equation (319) renders

$$r \frac{\partial^2 F_1}{\partial z^2} + \frac{1}{r} \frac{\partial^2 F_1}{\partial \varphi^2} - \frac{\partial F^2}{\partial z \partial r} + \frac{\partial}{\partial r}\left[r \frac{\partial F_1}{\partial r} + \frac{\partial F_2}{\partial z}\right] = 0$$

Division by r gives

$$\frac{\partial^2 F_1}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2 F_1}{\partial \varphi^2} + \frac{1}{r} \frac{\partial}{\partial r}\left[r \frac{\partial F_1}{\partial r}\right] = 0 \qquad (321)$$

F_2:

Partial differentiation of equation (307) with respect to z gives

$$\frac{\partial^2 F_1}{\partial z \partial r} + \frac{1}{r} \frac{\partial^2 F_2}{\partial z^2} + \frac{\partial^2 F_3}{\partial z \partial \varphi} = 0 \qquad (322)$$

Partial differentiation of equation (309) with respect to r gives

$$\frac{\partial^2 F_1}{\partial z \partial r} - \frac{\partial}{\partial r}\left[\frac{1}{r} \frac{\partial F_2}{\partial r}\right] - \frac{\partial}{\partial r}\left[\frac{1}{r} \frac{\partial F_4}{\partial \varphi}\right] = 0 \qquad (323)$$

Subtraction of equation (323) fron equation (322) renders

$$\frac{1}{r} \frac{\partial^2 F_2}{\partial z^2} + \frac{\partial}{\partial r}\left[\frac{1}{r} \frac{\partial F_2}{\partial r}\right] + \frac{\partial^2 F_3}{\partial z \partial \varphi} + \frac{\partial}{\partial r}\left[\frac{1}{r} \frac{\partial F_4}{\partial \varphi}\right] = 0$$

i.e.,

$$\frac{1}{r}\frac{\partial^2 F_2}{\partial z^2} + \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial F_2}{\partial r}\right] + \frac{\partial}{\partial \varphi}\left[\frac{\partial F_3}{\partial z} + \frac{\partial}{\partial r}\left(\frac{F_4}{r}\right)\right] = 0 \tag{324}$$

From equation (316), rearrangement gives

$$\frac{\partial F_3}{\partial z} = \frac{1}{r^3}\frac{\partial F_2}{\partial \varphi} - \frac{1}{r^2}\frac{\partial (r\, F_4)}{\partial r} \tag{325}$$

Substituting for $\frac{\partial F_3}{\partial z}$ from equation (325) in equation (324) and multiplying the resulting expression by r gives

$$\frac{\partial^2 F_2}{\partial z^2} + r\,\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial F_2}{\partial r}\right] + \frac{\partial}{\partial \varphi}\left[\frac{1}{r^2}\frac{\partial F_2}{\partial \varphi} - \frac{1}{r}\frac{\partial (r\, F_4)}{\partial r} + r\,\frac{\partial}{\partial r}\left(\frac{F_4}{r}\right)\right] = 0$$

i.e.,

$$\frac{\partial^2 F_2}{\partial z^2} + r\,\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F_2}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 F_2}{\partial \varphi^2}$$

$$= \frac{\partial}{\partial \varphi}\left[\frac{1}{r}\frac{\partial (r\, F_4)}{\partial r} - r\,\frac{\partial}{\partial r}\left(\frac{F_4}{r}\right)\right]$$

$$= \frac{\partial}{\partial \varphi}\left[\frac{1}{r}\left(F_4 + r\,\frac{\partial F_4}{\partial r}\right) - r\left(\frac{1}{r}\frac{\partial F_4}{\partial r} - \frac{1}{r^2}\,F_4\right)\right]$$

$$= \frac{\partial}{\partial\varphi}\left[\frac{F_4}{r} + \frac{\partial F_4}{\partial r} - \frac{\partial F_4}{\partial r} + \frac{1}{r} F_4\right]$$

$$= \frac{\partial}{\partial\varphi}\left(\frac{2\ F_4}{r}\right) = \frac{2}{r}\ \frac{\partial F_4}{\partial\varphi} \tag{326}$$

From equation (309),

$$\frac{1}{r}\ \frac{\partial F_4}{\partial\varphi} = \frac{\partial F_1}{\partial z} - \frac{1}{r}\ \frac{\partial F_2}{\partial r}$$

The equation (326), after rearrangement, becomes

$$\frac{\partial^2 F_2}{\partial z^2} + r\ \frac{\partial}{\partial r}\left(\frac{1}{r}\ \frac{\partial F_2}{\partial r}\right) + \frac{1}{r^2}\ \frac{\partial^2 F_2}{\partial\varphi^2} + \frac{2}{r}\ \frac{\partial F_2}{\partial r} = 2\ \frac{\partial F_1}{\partial z}$$

i.e.,

$$\frac{\partial^2 F_2}{\partial z^2} + \frac{1}{r^2}\ \frac{\partial^2 F_2}{\partial\varphi^2} + \left[r\ \frac{\partial}{\partial r}\left(\frac{1}{r}\ \frac{\partial F_2}{\partial r}\right) + \frac{2}{r}\ \frac{\partial F_2}{\partial r}\right] = 2\ \frac{\partial F_1}{\partial z} \tag{327}$$

Now

$$r\ \frac{\partial}{\partial r}\left(\frac{1}{r}\ \frac{\partial F_2}{\partial r}\right) + \frac{2}{r}\ \frac{\partial F_2}{\partial r}$$

$$= r\left(-\frac{1}{r^2}\ \frac{\partial F_2}{\partial r} + \frac{1}{r}\ \frac{\partial^2 F_2}{\partial r}\right) + \frac{2}{r}\ \frac{\partial F_2}{\partial r}$$

$$= \frac{1}{r} \frac{\partial F_2}{\partial r} + \frac{\partial^2 F_2}{\partial r} = \frac{1}{r}\left(\frac{\partial F_2}{\partial r} + r \frac{\partial^2 F_3}{\partial r}\right)$$

$$= \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial F_2}{\partial r}\right) \qquad (328)$$

Use of equation (328) in equation (328) gives

$$\frac{\partial^2 F_2}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2 F_2}{\partial \varphi^2} + \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial F_2}{\partial r}\right) = 2 \frac{\partial F_1}{\partial z} \qquad (329)$$

F_3:

Multiplying equation (308) by r and then partially differentiating the resulting expression with respect to r gives

$$\frac{\partial^2 F_1}{\partial r \partial \varphi} - \frac{\partial}{\partial r}\left[r \frac{\partial}{\partial r}\left(r F_3\right)\right] + \frac{\partial^2 \left(r F_4\right)}{\partial r \partial z} = 0 \qquad (330)$$

Partial differentiation of equation (316) with respect to z renders

$$\frac{1}{r} \frac{\partial^2 F_2}{\partial z \partial \varphi} - r^2 \frac{\partial^2 F_3}{\partial z^2} - \frac{\partial^2 \left(r F_4\right)}{\partial z \partial r} = 0 \qquad (331)$$

Summation of equations (330) and (331) gives

$$\frac{\partial}{\partial \varphi}\left[\frac{\partial F_1}{\partial r} + \frac{1}{r} \frac{\partial F_2}{\partial z}\right] - \frac{\partial}{\partial r}\left[r \frac{\partial}{\partial r}\left(r F_3\right)\right] - r^2 \frac{\partial^2 F_3}{\partial z^2} = 0 \qquad (332)$$

From equation (307),

$$\frac{\partial F_1}{\partial r} + \frac{1}{r}\,\frac{\partial F_2}{\partial z} = 1\,\frac{\partial F_3}{\partial \varphi}$$

Then equation (332), after multiplying by $-\frac{1}{r^2}$ and rearranging, becomes

$$\frac{\partial^2 F_3}{\partial z^2} + \frac{1}{r^2}\,\frac{\partial^2 F_3}{\partial \varphi^2} + \frac{1}{r^2}\,\frac{\partial}{\partial r}\left[r\,\frac{\partial}{\partial r}\left(r\,F_3\right)\right] = 0 \qquad (333)$$

F_4:

Partial differentiation of equation (308) with respect to z gives

$$\frac{\partial^2}{\partial z \partial \varphi}\left(\frac{F_1}{r}\right) - \frac{\partial^2}{\partial z \partial r}\left(r\,F_3\right) + \frac{\partial^2 F_4}{\partial z^2} = 0 \qquad (334)$$

Dividing equation (309) by $-r$ and partially differentiating the resulting expression with respect to φ, one obtains

$$-\frac{\partial^2}{\partial \varphi \partial z}\left(\frac{F_1}{r}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{r^2}\,\frac{\partial F_2}{\partial r}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{r^2}\,\frac{\partial F_4}{\partial \varphi}\right) = 0 \qquad (335)$$

Summation of equations (334) and (335) renders

$$\frac{\partial}{\partial \varphi}\left(\frac{1}{r^2}\,\frac{\partial F_2}{\partial r}\right) - \frac{\partial^2}{\partial z \partial r}\left(r\,F_3\right) + \frac{\partial^2}{\partial z^2}\,F_4 + \frac{\partial^2}{\partial \varphi^2}\left(\frac{F_4}{r^2}\right) = 0 \qquad (336)$$

From equation (316),

$$\frac{1}{r}\frac{\partial F_2}{\partial \varphi} = r^2\frac{\partial F_3}{\partial z} + \frac{\partial\left(r\ F_4\right)}{\partial r}$$

Multiplying by r and partially differentiating with respect to r

$$\frac{\partial^2 F_2}{\partial r \partial \varphi} = \frac{\partial}{\partial r}\left[r^3\ \frac{\partial F_3}{\partial z}\right] + \frac{\partial}{\partial r}\left[r\ \frac{\partial}{\partial r}\left(r\ F_4\right)\right]$$

Dividing by r^2 gives

$$\frac{\partial}{\partial \varphi}\left[\frac{1}{r^2}\ \frac{\partial F_2}{\partial r}\right] = \frac{1}{r^2}\ \frac{\partial}{\partial r}\left[r^3\ \frac{\partial F_3}{\partial z}\right] + \frac{1}{r^2}\ \frac{\partial}{\partial r}\left[r\ \frac{\partial}{\partial r}\left(r\ F_4\right)\right] \qquad (337)$$

Substituting for

$$\frac{\partial}{\partial \varphi}\left[\frac{1}{r^2}\ \frac{\partial F_2}{\partial r}\right]$$

from equation (337) in equation (336) gives

$$\frac{1}{r^2}\ \frac{\partial}{\partial r}\left[r^3\ \frac{\partial F_3}{\partial z}\right] - \frac{\partial^2}{\partial r \partial z}\left(r\ F_3\right) + \frac{\partial^2}{\partial z^2}F_4 + \frac{\partial^2}{\partial \varphi^2}\left(\frac{F_4}{r^2}\right) +$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r\ \frac{\partial}{\partial r}\left(r\ F_4\right)\right] = 0$$

or, rearranging

$$\frac{\partial^2 F_4}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 F_4}{\partial \varphi^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(r F_4\right)\right]$$

$$= \frac{\partial^2}{\partial r \partial z}\left(r F_3\right) - \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3\frac{\partial F_3}{\partial z}\right]$$

$$= \frac{\partial^2}{\partial r \partial z}\left(r F_3\right) - \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial z}\left(r F_3\right)\right]$$

$$= \frac{\partial^2}{\partial r \partial z}\left(r F_3\right) - \frac{1}{r^2}\left[r^2\frac{\partial^2}{\partial r \partial z}\left(r F_3\right) + 2 r \frac{\partial}{\partial z}\left(r F_3\right)\right]$$

$$= - \frac{2 r \cdot r}{r^2}\frac{\partial F_3}{\partial z} = - 2 \frac{\partial F_3}{\partial z} \qquad (338)$$

Separation of the Variables

Separation of ξ's in the Form of Partial Differential Equation Containing F's:

Reference is made to equations (303) through (306).

Separation of ξ_1, the distortion in radial directions:

From equation (304),

$$\frac{\partial \xi_3}{\partial r} = - \frac{2 F_2}{r} + \frac{\partial \xi_1}{\partial z} \qquad (339)$$

From equation (305),

$$\frac{\partial}{\partial r}\left(r\,\xi_2\right) = -\,2\,F_3\,r^2 + \frac{\partial \xi_1}{\partial \varphi} \qquad (340)$$

Multiplying equation (303) by $\frac{1-2\nu}{1-\nu}$ gives

$$\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\xi_1\right) + \frac{\partial \xi_3}{\partial z} + \frac{\partial}{\partial \varphi}\left(\frac{\xi_2}{r}\right) = \frac{1-2\nu}{1-\nu}\,F_1$$

Differentiating with respect to r

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\xi_1\right)\right] + \frac{\partial^2 \xi_3}{\partial z\partial r} + \frac{\partial^2}{\partial \varphi\partial r}\left(\frac{\xi_2}{r}\right) = \frac{1-2\nu}{1-\nu}\,\frac{\partial F_1}{\partial r} \qquad (341)$$

Now

$$\frac{\partial^2 \xi_3}{\partial z\partial r} + \frac{\partial^2}{\partial \varphi\partial r}\left(\frac{\xi_2}{r}\right) = \frac{\partial}{\partial z}\left(\frac{\partial \xi_3}{\partial r}\right) + \frac{\partial}{\partial \varphi}\,\frac{\partial}{\partial r}\left(\frac{\xi_2 r}{r^2}\right)$$

$$= \frac{\partial}{\partial z}\left(\frac{\partial \xi_3}{\partial r}\right) + \frac{\partial}{\partial \varphi}\left[\xi_2 r\left(-\,\frac{2}{r^3}\right) + \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(\xi_2 r\right)\right]$$

$$= \frac{\partial}{\partial z}\left(\frac{\partial \xi_3}{\partial r}\right) + \frac{\partial}{\partial \varphi}\left(-\,\frac{2\,\xi_2}{r^2}\right) + \frac{1}{r^2}\,\frac{\partial}{\partial \varphi}\,\frac{\partial}{\partial r}\left(\xi_2 r\right) \qquad (342)$$

Using equations (339) and (340), equation (342) would become

$$\frac{\partial^2 \xi_3}{\partial z\partial r} + \frac{\partial^2}{\partial \varphi\partial r}\left(\frac{\xi_2}{r}\right) = \frac{\partial}{\partial z}\left[\frac{\partial \xi_1}{\partial z} - \frac{2F_2}{r}\right] - \frac{2}{r^2}\,\frac{\partial \xi_2}{\partial \varphi} +$$

$$+ \frac{1}{r^2} \frac{\partial}{\partial \varphi} \left[- 2 F_3 r^2 + \frac{\partial \xi_1}{\partial \varphi} \right]$$

$$= \frac{\partial^2 \xi_1}{\partial z^2} - \frac{2}{r} \frac{\partial F_2}{\partial z} - \frac{2}{r^2} \frac{\partial \xi_2}{\partial \varphi} - 2 \frac{\partial F_3}{\partial \varphi} + \frac{1}{r^2} \frac{\partial^2 \xi_1}{\partial \varphi^2} \qquad (343)$$

Subtracting equation (343) from equation (341)

$$\frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \xi_1 \right) \right] + \frac{\partial^2 \xi_1}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2 \xi_1}{\partial \varphi^2}$$

$$= \frac{1 - 2\nu}{1 - \nu} \frac{\partial F_1}{\partial r} + \frac{2}{r} \frac{\partial F_2}{\partial z} + 2 \frac{\partial F_3}{\partial \varphi} + \frac{2}{r^2} \frac{\partial \xi_2}{\partial \varphi} \qquad (344)$$

Separation of ξ_2 , the distortion in tangential direction yields:

From equation (305),

$$\frac{\partial \xi_1}{\partial \varphi} = \frac{\partial}{\partial r} \left(r \xi_2 \right) + 2 r^2 F_3$$

or

$$\frac{\partial}{\partial \varphi} \left(r \xi_1 \right) = r \frac{\partial}{\partial r} \left(r \xi_2 \right) + 2 r^3 F_3 \qquad (345)$$

From equation (306),

$$\frac{\partial \xi_3}{\partial \varphi} = r \left[- 2 F_4 + \frac{\partial \xi_2}{\partial z} \right] \qquad (346)$$

Multiplying equation (303) by $\frac{1 - 2\nu}{1 - \nu}$ and then partially differentiating with respect to φ ,

$$\frac{\partial}{\partial\varphi}\left[\frac{1}{r}\ \frac{\partial}{\partial r}\left(r\ \xi_1\right)\right] + \frac{\partial^2 \xi_3}{\partial\varphi\partial z} + \frac{1}{r}\ \frac{\partial^2 \xi}{\partial\varphi} = \frac{1 - 2\nu}{1 - \nu}\ \frac{\partial F_1}{\partial\varphi}$$

Multiplying (r) results in

$$\frac{\partial}{\partial\varphi}\left[\frac{\partial}{\partial r}\left(r\ \xi_1\right)\right] + r\ \frac{\partial}{\partial z}\left(\frac{\partial \xi_3}{\partial\varphi}\right) + \frac{\partial^2 \xi_2}{\partial\varphi^2} = \frac{1 - 2\nu}{1 - \nu}\ \frac{\partial F_1}{\partial\varphi}\ r \qquad (347)$$

Using equations (345) and (346) in equation (347) gives

$$\frac{\partial}{\partial r}\left[r\ \frac{\partial}{\partial r}\left(r\ \xi_2\right) + 2\ r^3\ F_3\right] + r\ \frac{\partial}{\partial z}\left[r\ \left(-\ 2\ F_4 + \frac{\partial \xi_2}{\partial z}\right)\right] + \frac{\partial^2 \xi_2}{\partial\varphi^2}$$

$$= r\ \frac{1 - 2\nu}{1 - \nu}\ \frac{\partial F_1}{\partial\varphi}$$

Rearrangement renders

$$\frac{\partial}{\partial r}\left[r\ \frac{\partial}{\partial r}\left(r\ \xi_2\right)\right] + r^2\ \frac{\partial^2 \xi_2}{\partial z^2} + \frac{\partial^2 \xi_2}{\partial\varphi^2}$$

$$= 2\ r^2\ \frac{\partial F_4}{\partial z} + \frac{1 - 2\nu}{1 - \nu}\ r\ \frac{\partial F_1}{\partial\varphi} - 2\ \frac{\partial}{\partial r}\left(r^3\ F_3\right) \qquad (348)$$

Separation of ξ_3 , the distortion in axial direction results:

From equation (304),

$$\frac{\partial \xi_1}{\partial z} = \frac{\partial \xi_3}{\partial r} + \frac{2\,F_2}{r} \tag{349}$$

From equation (306),

$$\frac{\partial \xi_2}{\partial z} = 2\,F_4 + \frac{1}{r}\,\frac{\partial \xi_3}{\partial \varphi} \tag{350}$$

From equation (303, by multiplying by $\frac{1-2\nu}{1-\nu}\,r$ one obtains

$$\frac{\partial}{\partial r}\left(r\,\xi_1\right) + r\,\frac{\partial \xi_3}{\partial z} + \frac{\partial \xi_2}{\partial \varphi} = \frac{1-2\nu}{1-\nu}\,F_1 r$$

Partial differentiation with respect to z gives

$$\frac{\partial}{\partial r}\left[r\,\frac{\partial \xi_1}{\partial z}\right] + r\,\frac{\partial^2 \xi_3}{\partial z^2} + \frac{\partial^2 \xi_2}{\partial z\,\partial \varphi} = \frac{1-2\nu}{1-\nu}\,r\,\frac{\partial F_1}{\partial z} \tag{351}$$

Use of equations (349) and (350) in equation (351),

$$\frac{\partial}{\partial r}\left[r\left(\frac{\partial \xi_3}{\partial r} + \frac{2\,F_2}{r}\right)\right] + r\,\frac{\partial^2 \xi_3}{\partial z^2} + \frac{\partial}{\partial \varphi}\left(2\,F_4 + \frac{1}{r}\,\frac{\partial \xi_3}{\partial \varphi}\right) = \frac{1-2\nu}{1-\nu}\,r\,\frac{\partial F_1}{\partial z}$$

Multiplying by $\frac{1}{r}$

$$\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial \xi_3}{\partial r}\right) + \frac{1}{r}\,\frac{\partial}{\partial r}\left(2\,F_2\right) + \frac{\partial^2 \xi_3}{\partial z^2} + \frac{2}{r}\,\frac{\partial F_4}{\partial \varphi} + \frac{1}{r^2}\,\frac{\partial^2 \xi_3}{\partial \varphi^2} = \frac{1-2\nu}{1-\nu}\,\frac{\partial F_1}{\partial z} \tag{352}$$

Rearranging

$$\frac{\partial^2 \xi_3}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \xi_3}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 \xi_3}{\partial \varphi^2} = \frac{1-2\nu}{1-\nu} \frac{\partial F_1}{\partial z} - \frac{2}{r}\left(\frac{\partial F_2}{\partial r} + \frac{\partial F_4}{\partial \varphi}\right)$$

SOLUTION FOR ξ_2

Assuming that the F's are zeros, equation (348) becomes

$$\frac{\partial}{\partial r}\left[r \frac{\partial}{\partial r}\left(r\ \xi_2\right)\right] + r^2 \frac{\partial^2 \xi_2}{\partial z^2} + \frac{\partial^2 \xi_2}{\partial \varphi^2} = 0$$

i.e.,

$$\frac{\partial}{\partial r}\left[r\ \xi_2 + r^2 \frac{\partial \xi_2}{\partial r}\right] + r^2 \frac{\partial^2 \xi_2}{\partial z^2} + \frac{\partial^2 \xi_2}{\partial \varphi^2} = 0$$

i.e.,

$$\xi_2 + r \frac{\partial \xi_2}{\partial r} + 2r \frac{\partial \xi_2}{\partial r} + r^2 \frac{\partial^2 \xi_2}{\partial r^2} + r^2 \frac{\partial^2 \xi_2}{\partial z^2} + \frac{\partial^2 \xi_2}{\partial \varphi^2} = 0 \qquad (353)$$

Let

$$\xi_2 = R_2(r)\ \phi_2(\varphi)\ Z_2(z) \qquad (354)$$

Then equation (353) becomes

$$R_2\ \phi_2\ Z_2 + 3r\ R_2'\ \phi_2\ Z_2 + r^2\ R_2''\ \phi_2\ Z_2 + r^2\ R_2\ \phi_2\ Z_2'' + R_2\ \phi_2''\ Z_2 = 0 \qquad (355)$$

Dividing equation (355) by ξ_2

$$1 + 3r \frac{R_2'}{R_2} + r^2 \frac{R_2''}{R_2} + r^2 \frac{Z_2''}{Z_2} + \frac{\phi_2''}{\phi_2} = 0$$

or

$$r^2 \frac{R_2''}{R_2} + 3r \frac{R_2'}{R_2} + r^2 \frac{Z_2''}{Z_2} + 1 = -\frac{\phi_2''}{\phi_2} = \lambda_2^2 \tag{356}$$

where λ_2 is a constant independent of r , z , and φ .

$$\phi_2'' + \lambda_2^2 \phi_2 = 0 \tag{357}$$

and

$$r^2 \frac{R_2''}{R_2} + 3r \frac{R_2'}{R_2} + r^2 \frac{Z_2''}{Z_2} + 1 = +\lambda_2^2 \tag{358}$$

The last equation can be rewritten as

$$\frac{R''}{R_2} + \frac{3}{r} \frac{R_2'}{R} + \frac{1}{r^2}\left(1 - \lambda_2^2\right) = \frac{Z_2''}{Z_2} = n^2 \tag{359}$$

where n is a general constant independent of both r and z . Equation (359) may be separated as follows:

$$Z_2'' + \mu^2 Z_2 = 0 \tag{360}$$

and

$$R_2'' + \frac{3}{r} R_2' + \frac{R_2}{r^2}\left(1 - \lambda_2^2\right) - \mu^2 R_2 = 0 \tag{361}$$

The general solution of equation (357) is

$$\phi_2(\varphi) = B_1 \cos\left(\lambda_2\varphi + \varphi_o\right) \tag{362}$$

where φ_o is the phase angle and B_1 an arbitrary constant.

In the case of six symmetrically spaced fibers surrounding a central fiber it is expected that

$$\phi_2(\varphi) = \phi_2\left(\varphi + \frac{j\pi}{3}\right) \tag{363}$$

where j is any integer.

This means that

$$\begin{aligned} \cos \lambda_2\varphi + \varphi_o &= \cos\left\{\lambda_2\left(\varphi + \frac{j\pi}{3}\right) + \varphi_o\right\} \\ &= \cos\left\{\lambda_2\varphi + \varphi_o + \frac{j\pi\,\lambda_2}{3}\right\} \end{aligned} \tag{364}$$

For equation (364) to be satisfied for any m ,

$$\frac{\pi\,\lambda_2}{3} = 2\,\pi\,k$$

i.e.,

$$\lambda_2 = 6\ k \tag{365}$$

where k is any integer.

Therefore, equation (362) becomes

$$\phi_2 = B_1 \cos\left(6\ k\varphi + \varphi_o\right) \tag{366}$$

and equation (361) becomes, after dividing by μ^2,

$$\frac{1}{\mu^2}\ \frac{d^2R_2}{dr^2} + \frac{3}{r\mu^2}\ \frac{dR_2}{dr} - R_2\left(\frac{36\ k^2 - 1}{\mu^2 r^2} + 1\right) = 0 \tag{367}$$

Let

$$ir\mu = x \tag{368}$$

Then

$$\frac{1}{r\mu^2}\ \frac{dR_2}{dr} = \frac{1}{r\mu^2}\ \frac{dR_2}{dx}\ \frac{dx}{dr} = \frac{1}{r\mu^2}\ \frac{dR_2}{dx}\ (i\mu)$$

$$= -\frac{1}{ir\mu}\ \frac{dR_2}{dx} = -\frac{1}{x}\ \frac{dR_2}{dx} \tag{369}$$

$$\frac{1}{\mu^2}\ \frac{d^2R_2}{dr^2} = \frac{1}{\mu^2}\ \frac{d}{dx}\left\{\frac{dR_2}{dx}\ \frac{dx}{dr}\right\}\frac{dx}{dr} = \frac{(i\mu)^2}{\mu^2}\ \frac{d^2R_2}{dx^2} = -\frac{d^2R_2}{dx^2} \tag{370}$$

Using equations (368), (369), and (370) in equation (367), and multiplying the resulting expression by -1 , yields

$$\frac{d^2R_2}{dx^2} + \frac{3}{x}\frac{dR_2}{dx} + \left(1 - \frac{36\,k^2 - 1}{x^2}\right)R_2 = 0 \tag{371}$$

To transform the last equation to a Bessel form,

$$R_2 = x^{\theta}\, y\,(x) \tag{372}$$

Then

$$\frac{dR_2}{dx} = \theta\, x^{\theta-1}\, y(x) + x^{\alpha}\, y'(x) \tag{373}$$

$$\frac{d^2R_2}{dx^2} = \theta\,(\theta-1)\, x^{\theta-2}\, y(x) + 2\theta\, x^{\theta-1}\, y'(x) + x^{\alpha}\, y''(x) \tag{374}$$

Using equations (372) through (374) in equation (371) and dividing the resulting expression by x^{θ} results in

$$y''(x) + \frac{2\theta + 3}{x}\, y'(x) + \left(1 - \frac{36\,k^2 - 1}{x^2}\right)y + \{3\alpha + \alpha\,(\alpha-1)\}\,\frac{y}{x^2} = 0 \tag{375}$$

Rewriting equation (375),

$$y''(x) + \frac{2\theta + 3}{x}\, y'(x) + \left\{1 - \frac{36\,k^2 - 1 - [3\theta + \theta\,(\theta-1)]}{x^2}\right\}y \tag{376}$$

θ is to be chosen so that

$$2\theta + 3 = 1$$

i.e.,

$$\theta = -1$$

Then equation (376) becomes

$$y'' + \frac{y'}{x} + \left[1 - \frac{(6\,k)^2}{x^2}\right] y = 0 \tag{377}$$

Equation (377) is a Bessel equation and

$$y = B_2\, J_{6k}(x) + B_3\, H^{(1)}_{6k}(x) \tag{378}$$

This is the general solution where B_2 and B_3 are arbitrary constants.

From equation (372),

$$R_2 = x^{\theta}\, y$$

Since $\theta = -1$,

$$R_2 = \frac{y}{x}$$

Substituting for y from equation (378)

$$R_2(x) = \frac{B_2}{x}\, J_{6k}(x) + \frac{B_3}{x}\, H^{(1)}_{6k}(x) \tag{379}$$

Because $x = i\mu r$, equation (379) is

$$R_2(r) = \frac{1}{\mu r}\left[\frac{B_2}{i} J_{6k}(i\mu r) + \frac{B_3}{i} H_{6k}^{(1)}(i\mu r)\right] \qquad (380)$$

The general solution of equation (360) is

$$Z_2 = B_4 \sin\left(\mu z + z_o\right) \qquad (381)$$

where B_4 and z_o are arbitrary constants.

Substituting for $R_2(r)$, $\phi_2(\varphi)$, and $Z_2(z)$ from equations (380), (366), and (381) respectively in equation (354)

$$\xi_2(r, \varphi, z) = \frac{1}{\mu r}\left[\frac{B_2}{i} J_{6k}(i\mu r) + \frac{B_3}{i} H_{6k}^{(1)}(i\mu r)\right] \times$$

$$\left[B_1 \cos\left(6k\varphi + \varphi_o\right)\right]\left[B_4 \sin\left(\mu z + z_o\right)\right]$$

$$= \frac{1}{\mu r}\left[B_5 J_{6k}(i\mu r) + B_6 H_{6k}^{(1)}(i\mu r)\right] \times$$

$$\cos\left(6k\varphi + \varphi_o\right) \times \sin\left(\mu z + z_o\right) \qquad (382)$$

where

$$B_5 = \frac{B_1 B_4 B_2}{i}$$

and

$$B_6 = \frac{B_1 B_4 B_3}{i}$$

Note that μ may assume many values, dependent upon boundary conditions. Let $\mu = \mu_n$ where n varies from 1 to infinity and is an integer. As stated earlier, k varies from 0 to infinity and is an integer. A general assumption is that the constants B_5 , B_6 , φ_o , and z_o and corresponding ξ_2 in equation (382) would be different for different combinations of k and n . To indicate this, the said parameters would be properly subscripted. Then equation (382) would take the following form.

$$\xi_{2nk}(r, \varphi, z) = \frac{1}{\mu_n r}\left[B_{5nk}\, J_{6k}\left(i\mu_n r\right) + B_{6nk}\, H^{(1)}_{6k}\left(i\mu_n r\right)\right] \times$$

$$\cos\left(6k\varphi + \varphi_{onk}\right) \times \sin\left(\mu_n z + z_{onk}\right)$$

$$= R_{2nk}\, \Phi_{2nk}\, Z_{2nk} \qquad (383)$$

Now the total distortion of ξ_2 is a summation of all ξ_{2nk} given by equation (383), or

$$\xi_2(r, \varphi, z) = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \xi_{2nk}(r, \varphi, z)$$

$$= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{\mu_n r}\left[B_{5nk}\, J_{6k}\left(i\mu_n r\right) + B_{6nk}\, H^{(1)}_{6k}\left(i\mu_n r\right)\right] \times$$

$$\left[\cos\left(6k\varphi + \varphi_{onk}\right)\right] \times \left[\sin\left(\mu_n z + z_{onk}\right)\right] \qquad (384)$$

SOLUTION FOR ξ_3

Consider equation (352). F's are to be taken as zero.

$$\frac{\partial^2 \xi_3}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \xi_3}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 \xi_3}{\partial \varphi^2} = 0 \tag{385}$$

i.e.,

$$\frac{\partial^2 \xi_3}{\partial z^2} + \frac{\partial^2 \xi_3}{\partial r^2} + \frac{1}{r} \frac{\partial \xi_3}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \xi_3}{\partial \varphi^2} = 0 \tag{386}$$

Let

$$\xi_3(r)\ \phi_3(\varphi)\ Z_3(r) \tag{387}$$

Then equation (386) becomes, after dividing by ξ_3 and rearranging

$$R_3\ \phi_3\ Z_3'' + R_3''\ \phi_3\ Z_3 + \frac{1}{r} R_3'\ \phi_3\ Z_3 + \frac{1}{r^2} Z_3\ R_3\ \phi_3'' = 0$$

i.e.,

$$\frac{Z_3''}{Z_3} + \frac{R_3''}{R_3} + \frac{1}{r} \frac{R_3'}{R_3} + \frac{1}{r^2} \frac{\phi_3''}{\phi_3} = 0$$

$$\frac{R_3''}{R_3} + \frac{1}{r} \frac{R_3'}{R_3} + \frac{1}{r^2} \frac{\phi_3''}{\phi_3} = -\frac{Z_3''}{Z_3} = w^2 \tag{388}$$

where w is a constant eigenvalue, independent of r , φ , or z .

Equation (388) on separation gives the following two equations.

$$Z_3'' + w^2 Z_3 = 0 \tag{389}$$

and

$$\frac{R_3''}{R_3} + \frac{1}{r}\frac{R_3'}{R_3} + \frac{1}{r^2}\frac{\phi_3''}{\phi_3} = w^2 \tag{390}$$

Equation (390) is

$$r^2 \frac{R_3''}{R_3} + r\frac{R_3'}{R_3} - w^2 r^2 = -\frac{\phi_3''}{\phi_3} = \lambda_3^2 \tag{391}$$

where λ_3 is a constant. Equation (391), on separation, gives

$$\phi_3'' + \lambda_3^2 \phi_3 = 0 \tag{392}$$

and

$$r^2 \frac{d^2R_3}{dr^2} + r\frac{dR_3}{dr} + \left(-w^2 r^2 - \lambda_3^2\right) R_3 = 0 \tag{393}$$

Let

$$x = irw \tag{394}$$

Then

$$r\frac{dR_3}{dr} = \frac{x}{iw}\frac{dR_3}{dx}\frac{dx}{dr} = \frac{x}{iw}\frac{dR_3}{dx}(iw) = x\frac{dR_3}{dx} \tag{395}$$

Similarly

$$r^2 \frac{d^2R_3}{dr^2} = x^2 \frac{d^2R_3}{dx^2} \tag{396}$$

Then equation (393) becomes

$$x^2 \frac{d^2R_3}{dx^2} + x \frac{dR_3}{dx} + \left(x^2 - \lambda_3^2\right) R_3 = 0 \tag{397}$$

Equation (397) is a Bessel equation and its general solution is

$$R_3(x) = C_1 J_{\lambda_3}(x) + C_2 H^{(1)}_{\lambda_3}(x) \tag{398}$$

A similar argument can be utilised in arriving at a solution, from equation (366), for equation (357). Then one obtains, with $\lambda_3 = 6g$, a solution for equation (392).

$$\phi_3 = C_3 \cos\left(6g\varphi + \varphi_o\right) \tag{399}$$

where g is any integer and φ_o an arbitrary constant.

Noting that $x = irw$ and $\lambda_3 = 6g$, equation (398) becomes

$$R_3(r) = C_1 J_{6g}(irw) + C_2 H^{(1)}_{6g}(irw) \tag{400}$$

The general solution of equation (389) is

$$Z_3(z) = C_4 \sin\left(wz + z_o\right) \tag{401}$$

where z_o is an arbitrary constant.

Substituting for $R_3(r)$, $\phi_3(\varphi)$, and $Z_3(z)$ from equations (400), (399), and (401) respectively in equation (387)

$$\xi_3(r,\varphi,z) = \left[C_5\ J_{6g}(irw) + C_6\ H_{6g}^{(1)}(irw)\right]\left[\cos\left(6g\varphi + \varphi_o\right)\right]\left[\sin\left(wz + z_o\right)\right] \quad (402)$$

where

$$C_5 = C_1\ C_3\ C_4$$

and

$$C_6 = C_2\ C_3\ C_4$$

Note that w may assume many values, depending upon the boundary conditions. Let $w = w_m$ where m varies from 1 to infinity and is an integer. The constants C_5 , C_6 , φ_o , and z_o and corresponding ξ_3 would be, in general, different for different combinations of g and m . To indicate this, proper subscripts would be used, giving equation (402) the following form:

$$\xi_{3mg} = \left[C_{5mg}\ J_{6k}\left(irw_m\right) + C_{6mg}\ H_{6g}^{(1)}\left(irw_m\right)\right]\left[\cos\left(6g\varphi + \varphi_{omg}\right)\right]\left[\sin\left(w_m z + z_{omg}\right)\right] \quad (403)$$

The total ξ_3 is a summation of all ξ_{3mg}'s given by equation (403):

$$\xi_3(r,\varphi,z) = \sum_{m=1}^{\infty}\sum_{g=0}^{\infty} \xi_{3mg}(r,\varphi,z) = \sum_{m=1}^{\infty}\sum_{g=0}^{\infty}\left[C_{5mg}\ J_{6g}\left(irw_m\right) + \right.$$

$$+ C_{6mg} H_{6g}^{(1)}\left(irv_m\right)\Big]\Big[\cos\left(6g\varphi + \varphi_{omg}\right)\Big]\Big[\sin\left(v_m z + z_{omg}\right)\Big] \tag{404}$$

COMPLEMENTARY SOLUTIONS FOR ξ_1

Taking F's as zero, equation (344) becomes,

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\xi_1\right)\right] + \frac{\partial^2 \xi_1}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 \xi_1}{\partial \varphi^2} = \frac{2}{r^2}\frac{\partial \xi_2}{\partial \varphi} \tag{405}$$

After simplification, equation (405) becomes

$$\frac{\partial^2 \xi_1}{\partial z^2} + \frac{\partial^2 \xi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \xi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \xi_1}{\partial \varphi^2} - \frac{\xi_1}{r^2} = \frac{2}{r^2}\frac{\partial \xi_2}{\partial \varphi} \tag{406}$$

Complementary function:

Consider

$$\frac{\partial^2 \xi_1}{\partial z^2} + \frac{\partial^2 \xi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \xi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \xi_1}{\partial \varphi^2} - \frac{\xi_1}{r^2} = 0 \tag{407}$$

Let

$$\xi_1 = R_1(r)\,\phi_1(\varphi)\,Z_1(z) \tag{408}$$

be the solution of equation (407). Substituting from equation (408) for ξ_1 in equation (407) and dividing the result by ξ_1

$$\frac{Z_1''}{Z_1} + \frac{R_1''}{R_1} + \frac{R_1'}{r\,R_1} + \frac{\phi_1''}{r^2\phi_1} - \frac{1}{r^2} = 0 \tag{409}$$

Readjusting,

$$\frac{R_1''}{R_1} + \frac{R_1'}{r\,R_1} + \frac{\phi_1''}{r^2\phi_1} - \frac{1}{r^2} = -\frac{Z_1''}{Z_1} = +\beta \tag{410}$$

where β is constant. Separating equation (410), the following two equations result:

$$Z_1'' + \beta^2 Z_1 = 0 \tag{411}$$

$$\frac{R_1''}{R_1} + \frac{R_1'}{r\,R_1} + \frac{\phi_1''}{r^2\phi_1} - \frac{1}{r^2} - \beta^2 = 0 \tag{412}$$

Consider equation (412). Multiplication by r^2 gives

$$\frac{r^2R_1''}{R_1} + \frac{r\,R_1'}{R_1} - 1 - \beta^2 r^2 = -\frac{\phi_1''}{\phi_1} = \lambda_1^2 \tag{413}$$

or

$$\phi_1'' + \lambda_1^2\,\phi_1 = 0 \tag{414}$$

and

$$\frac{r^2 R_1''}{R_1} + \frac{r\,R_1'}{R_1} - 1 - \beta^2 r^2 - \lambda_1^2 = 0 \tag{415}$$

Multiplying equation (415) by R_1

$$r^2 R_1'' + r\,R_1' - R_1 - \beta^2 r^2 R_1 - \lambda_1^2 R_1 = 0 \tag{416}$$

Let

$$x = i\beta r \tag{417}$$

Then

$$r\frac{dR_1}{dr} = \frac{x}{i\beta}\frac{dR_1}{dx}\frac{dx}{dr} = x\frac{dR_1}{dx}\quad r^2\frac{d^2R_1}{dr^2}$$

$$= \frac{x^2}{(i\beta)^2}\frac{d}{dr}\left[\frac{dR_1}{dr}\right] = \frac{x^2}{(i\beta)^2}\frac{d}{dx}\left[\frac{dR_1}{dx}\frac{dx}{dr}\right]$$

$$= x^2\frac{d^2R_1}{dx^2} \quad \beta^2 r^2 = i^2\beta^2 r^2 = x^2$$

Then equation (416) becomes

$$x^2\frac{d^2R_1}{dx^2} + x\frac{dR_1}{dx} + \left(x^2 - \left(\lambda_1^2 + 1\right)\right) R_1 = 0 \tag{418}$$

The solution of equation (411) is

$$Z_1 = A_1 \sin\left(\beta_z + z_o\right) \tag{419}$$

The solution of equation (414) is

$$\phi_1 = A_2 \cos\left(\lambda_1\varphi + \varphi_o\right) \tag{420}$$

where z_o and φ_o are arbitrary constants.

Note that

$$\phi_1(\varphi) = \phi_1\left(\varphi + \frac{j\pi}{3}\right) \tag{421}$$

where j is any integer, is expected, due to six-fiber symmetry around the central fiber. Then examination of equation (420) gives

$$\lambda_1 = 6\alpha \tag{422}$$

where α is an integer.

Now equation (420) becomes

$$\phi_1 = A_2 \cos\left(6\alpha\varphi + \varphi_o\right) \tag{423}$$

Equation (418) is a Bessel equation. Noting that $x = i\beta r$ from equation (417) and $\lambda_1 = 6k$ from equation (422), the general solution of equation (418) may be rewritten as

$$R_1(r) = A_3\, J_{\sqrt{36\alpha^2 + 1}}(i\beta r) + A_4\, H^{(1)}_{\sqrt{36\alpha^2 + 1}}(i\beta r) \tag{424}$$

Combining equations (408), (419), (423), and (424),

$$\xi_1(r,\varphi,z) = R_1(r)\ \phi_1(\varphi)\ Z_1(z)$$

$$= \left[A_5\ J_{\sqrt{36\alpha^2 + 1}}(i\beta r) + A_6\ H^{(1)}_{\sqrt{36\alpha^2 + 1}}(i\beta r)\right]\left[\cos\left(6\alpha\varphi + \varphi_o\right)\right]\left[\sin\left(\beta_z + z_o\right)\right] \qquad (425)$$

where $A_5 = A_3\ A_1\ A_2$ and $A_6 = A_4\ A_1\ A_2$.

Let β_p be the p^{th} value of β . Noting the dependence of A_5 , A_6 , φ_o , z_o and ξ_1 on p and α , proper subscripts would be used. Equation (425) then becomes

$$\xi_{1p\alpha}(r,\varphi,z) = \left[A_{5p\alpha}\ J_{\sqrt{36\alpha^2 + 1}}\left(i\beta_p r\right) + A_{6p\alpha}\ H^{(1)}_{\sqrt{36\alpha^2 + 1}}\left(i\beta_p r\right)\right]\left[\cos\left(6\alpha\varphi + \varphi_{op\alpha}\right)\right]\left[\sin\left(\beta_p z + z_{op\alpha}\right)\right] \qquad (426)$$

Summation of all $\xi_{1p\alpha}$ given by equation (426) would give the total ξ_1 .

$$\xi_1(r,\varphi,z) = \sum_{p=1}^{\infty}\sum_{\alpha=0}^{\infty} \xi_{1p\alpha}(r,\varphi,z) =$$

$$= \sum_{p=1}^{\infty} \sum_{\alpha=0}^{\infty} \left[A_{5p\alpha} J_{\sqrt{36\alpha^2+1}} \left(i\beta_p r \right) + \right.$$

$$\left. A_{6p\alpha} H^{(1)}_{\sqrt{36\alpha^2+1}} \left(i\beta_p r \right) \right] \left[\cos\left(6\alpha\varphi + \right.\right.$$

$$\left.\left. \varphi_{op\alpha} \right) \right] \left[\sin\left(\beta_p z + z_{op\alpha} \right) \right] \qquad (427)$$

For completeness, the other two ξ's will be stated herein.

$$\xi_2 = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{\mu_n r} \left[B_{5nk} J_{6k} \left(i\mu_n r \right) + \right.$$

$$\left. B_{6nk} H^{(1)}_{6k} \left(i\mu_n r \right) \right] \left[\cos\left(6k\varphi + \varphi_{onk} \right) \right] \left[\sin\left(\mu_n z + z_{onk} \right) \right]$$

$$\xi_3 = \sum_{m=1}^{\infty} \sum_{g=0}^{\infty} \left[C_{5mg} J_{6g} \left(irw_m \right) + \right.$$

$$\left. C_{6mg} H^{(1)}_{6g} \left(irw_m \right) \right] \left[\cos\left(6g\varphi + \varphi_{omg} \right) \right] \left[\sin\left(w_m z + z_{omg} \right) \right]$$

<u>Particular Integral for $\xi_1\left(\bar{\xi}_1\right)$</u>

The particular integral of equation (406) must be found. The right-hand side is $\frac{2}{r^2} \frac{\partial \xi_2}{\partial \varphi}$.

From equation (384),

$$\xi_2(r,\varphi,z) = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \xi_{2nk}(r,\varphi,z)$$

$$= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} R_{2nk}\, \Phi_{2nk}\, Z_{2nk} \qquad (428)$$

where
$$R_{2nk} = \frac{1}{\mu_n r}\left[B_{5nk}\, J_{6k}\left(i\mu_n r\right) + B_{6nk}\, H_{6k}^{(1)}\left(i\mu_n r\right)\right]$$

$$\Phi_{2nk} = \cos\left(6k\varphi + \varphi_{onk}\right)$$

$$Z_{2nk} = \sin\left(\mu_n z + z_{onk}\right)$$

Then the right-hand side of equation (406) would be

$$= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{2}{r^2}\, R_{2nk}\, Z_{2nk}\, \Phi_{2nk}'$$

$$= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} -\frac{12\, k\, R_{2nk}}{r^2}\, Z_{2nk} \sin\left(6k\varphi + \varphi_{onk}\right) \qquad (429)$$

Let

$$\bar{R}_{2nk} = -\frac{12\, k\, R_{2nk}}{r^2}$$

$$= - \frac{12\,k}{\mu_n r^3}\left[B_{5nk}\, J_{6k}\left(i\mu_n r\right) + B_{6nk}\, H_{6k}^{(1)}\left(i\mu_n r\right)\right] \tag{430}$$

and

$$\bar{\Phi}_{2nk} = \sin\left(6k\varphi + \varphi_{onk}\right) \tag{431}$$

Then the right-hand side of equation (429) becomes

$$= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \bar{R}_{2nk}\, Z_{2nk}\, \bar{\Phi}_{2nk} \tag{432}$$

Let

$$\bar{R}_{2nk}\, Z_{2nk}\, \bar{\Phi}_{2nk} = (\text{right-hand side})_{nk} \tag{433}$$

Then the right-hand side of equation (432) is

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} (\text{right-hand side})_{nk} \tag{434}$$

To obtain the desired particular integral, each $(\text{right-hand side})_{nk}$ would be taken as right-hand for equation (406). Particular integrals for such equations would be found and all such resulting expressions summed. For example,

$$\frac{\partial^2\left(\bar{\xi}_1\right)_{nk}}{\partial z^2} + \frac{\partial^2\left(\bar{\xi}_1\right)_{nk}}{\partial r^2} + \frac{1}{r}\,\frac{\partial\left(\bar{\xi}_1\right)_{nk}}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2\left(\bar{\xi}_1\right)_{nk}}{\partial \varphi^2} -$$

$$= \frac{\left(\bar{\xi}_1\right)_{nk}}{r^2} = (\text{right-hand side})_{nk} \tag{435}$$

where $\left(\bar{\xi}_1\right)_{nk}$ is the particular integral ξ_1 for the specific n,k combination.

Let

$$\left(\bar{\xi}_1\right)_{nk} = R_{nk}(r)\, \phi_{nk}(\varphi)\, Z_{nk}(z) \tag{436}$$

Substituting from equation (436) in equation (435) and rearranging results in

$$Z_{nk}(z)\, \phi_{nk}(\varphi) \left\{ \frac{Z''_{nk}(z)}{Z_{nk}(z)} R_{nk}(r) + R''_{nk}(r) + \frac{1}{r} R'_{nk}(r) + \frac{1}{r^2} \frac{\phi''_{nk}(\varphi)}{\phi(\varphi)} R_{nk}(r) - \frac{R_{nk}(r)}{r^2} \right\}$$

$$= \bar{R}_{2nk}\, Z_{2nk}\, \bar{\phi}_{2nk} \tag{437}$$

Equation (437) must be satisfied for any values of r, φ, and z. Functions separated in this manner on the two sides are to be separately equal. Equating parts in z and φ,

$$Z_{nk}(z) = Z_{2nk} = \sin\left(\mu_n z + z_{onk}\right) \tag{438}$$

and

$$\phi_{nk}(\varphi) = \bar{\phi}_{2nk} = \sin\left(6k\varphi + \varphi_{onk}\right) \tag{439}$$

giving

$$\frac{Z''_{nk}(z)}{Z_{nk}(z)} = -\mu_n^2$$

and

$$\frac{\phi''_{nk}(\varphi)}{\phi(\varphi)} = -(6k)^2$$

Then equating the parts in r on both of the sides of equation (430),

$$-\mu_n^2 R_{nk} + R''_{nk} + \frac{R'_{nk}}{r} - \frac{(6k)^2}{r^2} R_{nk} - \frac{R_{nk}}{r^2} = \bar{R}_{2nk} \tag{440}$$

Substituting for $\bar{R}_{2nk}$ from equation (430), equation (440) becomes

$$-\mu_n^2\left[-\frac{R''_{nk}}{\mu_n^2} - \frac{R'_{nk}}{\mu_n^2 r} + R_{nk}\left(1 + \frac{36k^2+1}{\mu_n^2 r^2}\right)\right]$$

$$= -\frac{12k}{\mu_n r^3}\left[B_{5nk} J_{6k}\left(i\mu_n r\right) + B_{6nk} H_{6k}^{(1)}\left(i\mu_n r\right)\right] \tag{441}$$

Let

$$i\mu_n r = \eta \tag{442}$$

Then equation (441) takes the following form.

$$\frac{d^2R_{nk}}{d\eta^2} + \frac{1}{\eta}\frac{dR_{nk}}{d\eta} + \left(1 - \frac{36k^2+1}{\eta}\right) R_{nk} =$$

$$= \frac{1}{r^3}\left[B_7\, J_{6k}(\eta) + B_8\, H^{(1)}_{6k}(\eta)\right] \qquad (443)$$

where

$$B_7 = \frac{12\, k\, B_{5nk}}{i} = -\,12\, ik\, B_{5nk} \qquad (444)$$

and

$$B_8 = \frac{12\, k\, B_{6nk}}{i} = -\,12\, ik\, B_{6nk} \qquad (445)$$

Let

$$R_{nk} = \bar{R}(\eta) = U(\eta)\, J_{\sqrt{36k^2+1}}(\eta) + V(\eta)\, H^{(1)}_{\sqrt{36k^2+1}}(\eta) \qquad (446)$$

be the particular integral of equation (443).

Then

$$\bar{R}'(\eta) = U'(\eta)\, J_{\sqrt{36k^2+1}}(\eta) + U(\eta)\, J'_{\sqrt{36k^2+1}}(\eta) +$$

$$V'(\eta)\, H^{(1)}_{\sqrt{36k^2+1}}(\eta) + V(\eta)\, H^{(1)'}_{6k}(\eta) \qquad (447)$$

Take

$$U'(\eta)\, J_{\sqrt{36k^2+1}} + V'(\eta)\, H^{(1)}_{\sqrt{36k^2+1}}(\eta) = 0 \qquad (448)$$

Then differentiating equation (447),

$$Y''(\eta) = U'(\eta)\, J'_{\sqrt{36k^2+1}}(\eta) + U(\eta)\, J''_{\sqrt{36k^2+1}}(\eta) +$$

$$V'(\eta)\, H^{(1)'}_{\sqrt{36k^2+1}}(\eta) + V(\eta)\, H^{(1)''}_{\sqrt{36k^2+1}}(\eta) \qquad (449)$$

Using equations (446) through (449) in equation (443) and rearranging gives

$$U(\eta)\left[J''_{\sqrt{36k^2+1}}(\eta) + \frac{1}{\eta} J'_{\sqrt{36k^2+1}}(\eta) + \right.$$

$$\left.\left(1 - \frac{36k^2+1}{\eta^2}\right) J_{\sqrt{36k^2+1}}(\eta)\right] + V(\eta)\left[H^{(1)''}_{\sqrt{36k^2+1}}(\eta) + \right.$$

$$\left.\frac{1}{\eta} H^{(1)'}_{\sqrt{36k^2+1}}(\eta) + \left(1 - \frac{36k^2+1}{\eta^2}\right) H^{(1)}_{\sqrt{36k^2+1}}(\eta)\right] +$$

$$U'(\eta)\, J'_{\sqrt{36k^2+1}}(\eta) + V'(\eta)\, H^{(1)'}_{\sqrt{36k^2+1}}(\eta)$$

$$= \frac{1}{\eta^3}\left[B_7\, J_{6k}(\eta) + B_8\, H^{(1)}_{6k}(\eta)\right] \qquad (450)$$

Noting that $J_{\sqrt{36k^2+1}}(\eta)$ and $H^{(1)}_{6k}(\eta)$ are solutions of the following Bessel equation,

$$y''(\eta) + \frac{y'(\eta)}{\eta} + \left(1 - \frac{36k^2+1}{\eta^2}\right) y = 0 \qquad (451)$$

results in

$$J''_{\sqrt{36k^2+1}}(\eta) + \frac{1}{\eta} J'_{\sqrt{36k^2+1}}(\eta) + \left(1 - \frac{36k^2+1}{\eta^2}\right) J_{\sqrt{36k^2+1}}(\eta) = 0 \qquad (452)$$

and

$$H^{(1)''}_{\sqrt{36k^2+1}}(\eta) + \frac{1}{\eta} H^{(1)'}_{\sqrt{36k^2+1}}(\eta) + \left(1 - \frac{36k^2+1}{\eta^2}\right) H^{(1)}_{\sqrt{36k^2+1}}(\eta) = 0 \qquad (453)$$

Using equations (452) and (453) in equation (454),

$$U'(\eta)\, J'_{\sqrt{36k^2+1}}(\eta) + V'(\eta)\, H^{(1)'}_{\sqrt{36k^2+1}}(\eta) = \frac{1}{\eta^3}\left[B_7\, J_{6k}(\eta) + B_8\, H^{(1)}_{6k}(\eta)\right] \qquad (454)$$

Solving for $U'(\eta)$ and $V'(\eta)$ from equations (448) and (454),

$$U'(\eta) = \frac{H^{(1)}_{\sqrt{36k^2+1}}(\eta)\left[B_7 J_{6k}(\eta) + B_8 H^{(1)}_{6k}(\eta)\right]}{\eta^3\left[J'_{\sqrt{36k^2+1}}(\eta)\, H^{(1)}_{\sqrt{36k^2+1}}(\eta) - J_{\sqrt{36k^2+1}}(\eta)\, H^{(1)'}_{\sqrt{36k^2+1}}(\eta)\right]} \tag{455}$$

and

$$V'(\eta) = -\frac{J_{\sqrt{36k^2+1}}(\eta)\left[B_7 J_{6k}(\eta) + B_8 H^{(1)}_{6k}(\eta)\right]}{\eta^3\left[J'_{\sqrt{36k^2+1}}(\eta)\, H^{(1)}_{\sqrt{36k^2+1}}(\eta) - J_{\sqrt{36k^2+1}}(\eta)\, H^{(1)'}_{\sqrt{36k^2+1}}(\eta)\right]} \tag{456}$$

because

$$J_p'(\eta)\, H_p^{(1)}(\eta) - J_p(\eta)\, H_p^{(1)'}(\eta) = \frac{2}{\pi i \eta} \tag{457}$$

Therefore, equations (455) and (456) become, respectively,

$$U'(\eta) = \frac{\pi i}{2}\,\frac{1}{\eta^2}\left\{H^{(1)}_{\sqrt{36k^2+1}}(\eta)\left[B_7 J_{6k}(\eta) + B_8 H^{(1)}_{6k}(\eta)\right]\right\} \tag{458}$$

and

$$V'(\eta) = -\frac{\pi i}{2}\,\frac{1}{\eta^2}\left\{J_{\sqrt{36k^2+1}}(\eta)\left[B_7 J_{6k}(\eta) + B_8 H^{(1)}_{6k}(\eta)\right]\right\} \tag{459}$$

Integrating the last two expressions with respect to η ,

$$U = \frac{\pi i}{2} \left(B_7 \, I_1 + B_8 \, I_2 \right) \tag{460}$$

and

$$V = - \frac{\pi i}{2} \left(B_7 \, I_3 + B_8 \, I_4 \right) \tag{461}$$

where
$$I_1 = \int \frac{H^{(1)}_{\sqrt{36k^2 + 1}}(\eta) \; J_{6k}(\eta)}{\eta^2} \, d\eta \tag{462}$$

$$I_2 = \int \frac{H^{(1)}_{\sqrt{36k^2 + 1}}(\eta) \; H^{(1)}_{6k}(\eta)}{\eta^2} \, d\eta \tag{463}$$

$$I_3 = \int \frac{J_{\sqrt{36k^2 + 1}}(\eta) \; J_{6k}(\eta)}{\eta^2} \, d\eta \tag{464}$$

$$I_4 = \int \frac{J_{\sqrt{36k^2 + 1}}(\eta) \; H^{(1)}_{6k}(\eta)}{\eta^2} \, d\eta \tag{465}$$

Equations (462) through (465) may be represented in general by the following:

$$I = \int \frac{Z_\gamma \, \bar{Z}_\delta \, d\eta}{\eta^2} \tag{466}$$

where

$$\left.\begin{array}{lll} Z \text{ and } \bar{Z} & \text{stand for} & H^{(1)} \text{ or } J \\ \gamma & \text{stand for} & \sqrt{36k^2 + 1} \\ \beta & \text{stand for} & 6k \\ Z_\gamma & \text{stand for} & Z_\gamma(\eta) \\ Z_\beta & \text{stand for} & Z_\beta(\eta) \end{array}\right\} \qquad (467)$$

Now

$$\frac{Z_\beta}{\eta} = \frac{Z_{\beta-1} + Z_{\beta+1}}{2\beta} \qquad (468)$$

Then equation (466) becomes

$$I = \frac{1}{2\beta}\left[\int \frac{Z_\gamma \bar{Z}_{\beta-1}}{\eta}\, d\eta + \int \frac{Z_\gamma Z_{\beta+1}}{\eta}\, d\eta\right. \qquad (469)$$

The following integration formula is to be noted:

$$\int \frac{1}{\eta} Z_p \bar{Z}_q \, d\eta = \eta \frac{Z_{p-1}\bar{Z}_q - Z_p \bar{Z}_{q-1}}{p^2 - q^2} - \frac{Z_p \bar{Z}_q}{p + q} \qquad (470)$$

Then

$$\int \frac{1}{\eta} Z_\gamma \bar{Z}_{\beta-1} \, d\eta = \eta \frac{Z_{\gamma-1}\bar{Z}_{\beta-1} - Z_\gamma \bar{Z}_{\beta-2}}{\gamma^2 - (\beta - 1)^2} - \frac{Z_\gamma \bar{Z}_{\beta-1}}{\gamma + \beta-1} \qquad (471)$$

and

$$\int \frac{1}{\eta} z_\gamma \, z_{\beta+1} \, d\eta = \eta \frac{z_{\gamma-1} \overline{z}_{\beta+1} - z_\gamma \overline{z}_\beta}{\gamma^2 - (\beta + 1)^2} - \frac{z_\gamma \overline{z}_{\beta+1}}{\gamma + \beta + 1}$$

Using equations (471) and (472) in equation (469) gives

$$I = \frac{1}{2\beta}\left[\eta\left(\frac{z_{\gamma-1} \overline{z}_{\beta-1} - z_\gamma \overline{z}_{\beta-2}}{\gamma^2 - (\beta - 1)^2} + \frac{z_{\gamma-1} \overline{z}_{\beta+1} - z_\gamma \overline{z}_\beta}{\gamma^2 - (\beta + 1)^2}\right) - z_\gamma\left(\frac{\overline{z}_{\beta-1}}{\gamma + \beta - 1} + \frac{\overline{z}_{\beta+1}}{\gamma + \beta + 1}\right)\right] \tag{473}$$

From equation (467), substituting for γ and β :

$$\gamma^2 - (\beta - 1)^2 = \left(\sqrt{36k^2 + 1}\right)^2 - (6k - 1)^2 = 12\ k$$

$$\gamma^2 - (\beta + 1)^2 = \left(\sqrt{36k^2 + 1}\right)^2 - (6k - 1)^2 = -\ 12\ k$$

Then the following are easily obtained:

$$\frac{z_{\gamma-1} \overline{z}_{\beta-1} - z_\gamma \overline{z}_{\beta-2}}{\gamma^2 - (\beta - 1)^2} + \frac{z_{\gamma-1} \overline{z}_{\beta+1} - z_\gamma \overline{z}_\beta}{\gamma^2 - (\beta + 1)^2} = \frac{z_{\gamma-1}\left[\overline{z}_{\beta-1} - \overline{z}_{\beta+1}\right] + z_\gamma\left[\overline{z}_\beta - \overline{z}_{\beta-2}\right]}{12\ k} \tag{474}$$

Using equation (474) in equation (473), then substituting for γ and β from equation (467) results in the following equation:

$$\begin{aligned} I = \frac{1}{12\,k}\Bigg\{\eta\, & \frac{Z_{\sqrt{36k^2+1}\,-1}\left[\bar{Z}_{6k-1} - \bar{Z}_{6k+1}\right] + Z_{\sqrt{36k^2+1}}\left[\bar{Z}_{6k} - \bar{Z}_{6k-2}\right]}{12\,k} - \\ & - Z_{\sqrt{36k^2+1}}\left(\frac{\bar{Z}_{6k-1}}{\sqrt{36k^2+1} + 6k - 1} + \frac{\bar{Z}_{6k+1}}{\sqrt{36k^2+1} + 6k + 1}\right)\Bigg\} \end{aligned} \qquad (475)$$

Equation (475) is equivalent to equation (466) with its integration accomplished. By analogy, similar expressions equivalent to equation (462) through (465) are as follows (in sequence).

$$\begin{aligned} I_1 = \frac{1}{12\,k}\Bigg\{\eta\, & \frac{H^{(1)}_{\sqrt{36k^2+1}\,-1}\left[J_{6k-1} - J_{6k+1}\right] + H^{(1)}_{\sqrt{36k^2+1}}\left[J_{6k} - J_{6k-2}\right]}{12\,k} - \\ & H^{(1)}_{\sqrt{36k^2+1}}\left[\frac{J_{6k-1}}{\sqrt{36k^2+1} + 6k - 1} + \frac{J_{6k+1}}{\sqrt{36k^2+1} + 6k + 1}\right]\Bigg\} \end{aligned} \qquad (476)$$

$$\begin{aligned} I_2 = \frac{1}{12\,k}\Bigg\{\eta\, & \frac{H^{(1)}_{\sqrt{36k^2+1}\,-1}\left[H^{(1)}_{6k-1} - H^{(1)}_{6k+1}\right] + H^{(1)}_{\sqrt{36k^2+1}}\left[H^{(1)}_{6k} - H^{(1)}_{6k-2}\right]}{12\,k} - \\ & - H^{(1)}_{\sqrt{36k^2+1}}\left[\frac{H^{(1)}_{6k-1}}{\sqrt{36k^2+1} + 6k - 1} + \frac{H^{(1)}_{6k+1}}{\sqrt{36k^2+1} + 6k + 1}\right]\Bigg\} \end{aligned} \qquad (477)$$

$$I_3 = \frac{1}{12\,k}\left\{\eta\,\frac{J_{\sqrt{36k^2+1}\,-1}\left[J_{6k-1} - J_{6k+1}\right] + J_{\sqrt{36k^2+1}}\left[J_{6k} - J_{6k-2}\right]}{12\,k} - \right.$$

$$\left. J_{\sqrt{36k^2+1}}\left[\frac{J_{6k-1}}{\sqrt{36k^2+1} + 6k - 1} + \frac{J_{6k+1}}{\sqrt{36k^2+1} + 6k + 1}\right]\right\} \tag{478}$$

$$I_4 = \left\{\frac{1}{12\,k}\;\eta\,\frac{J_{\sqrt{36k^2+1}\,-1}\left[H^{(1)}_{6k-1} - H^{(1)}_{6k+1}\right] + J_{\sqrt{36k^2+1}}\left[H^{(1)}_{6k} - H^{(1)}_{6k-2}\right]}{12\,k} - \right.$$

$$\left. J_{\sqrt{36k^2+1}}\left[\frac{H^{(1)}_{6k-1}}{\sqrt{36k^2+1} + 6k - 1} + \frac{H^{(1)}_{6k+1}}{\sqrt{36k^2+1} + 6k + 1}\right]\right\} \tag{479}$$

The total solution of ξ_1 is

$$\xi_1 = \xi_1(r,\varphi,s) + \bar{\xi}_1 \tag{480}$$

Equation (480) contains $\xi_1(r,\varphi,s)$, the solution of the homogeneous part of equation (406) given in equation (427). Using equation (436), the following is restated:

$$\bar{\xi}_{1nk} = R_{nk}(r)\;\phi_{nk}(\varphi)\;Z_{nk}(s) \tag{481}$$

$Z_{nk}(s)$ and $\phi_{nk}(\varphi)$ are given in equations (438) and (439), and restated herein:

$$Z_{nk}(s) = Z_{2nk} = \sin\left(\mu_n s + s_{onk}\right)$$

$$\phi_{nk}(\varphi) = \bar{\phi}_{2nk} = \sin\left(6k\varphi + \varphi_{onk}\right)$$

The most difficult value to establish is $R(r)$, as shown in the analysis contained in equations (440) through (479). The subsequent steps must be followed to obtain $R(r)$: (1) calculate first the Bessel expression in equations (476) through (479); (2) introduce the obtained I_1 to I_4 into equations (460) and (461) having, then, U and V. Entering this expression into equation (446), gives the values for $R_{nk}(r)$:

$$R_{nk} = \bar{R}(\eta) = U(\eta)\, J_{\sqrt{36k^2 + 1}}(\eta) + V(\eta)\, H_{\sqrt{36k^2 + 1}}(\eta)$$

Having the values of distortion $\xi_1\ \xi_2\ \xi_3$, then, the stress distribution is determined by equations (1) through (5) and equation (7) for σ.

Thus, the mathematical - physical fundamentals for the internal mechanics of parallel fibers are established.

Review of equations (438) and (439) clearly indicates that the ξ's are of an undulatory nature. But since the solutions are sums over several eigenvalues characterized by parameters n, they could be regarded as a Fourier series. The coefficients of the series obviously show a certain characteristic about a value n_o. The assumption can therefore be that the functions ξ consist primarily of those cosine functions whose order is grouped within a narrow region around this value n_o. The conclusion, then, can be that the functions ξ will consist, for the most part, of the functions

$$\cos \frac{n_o \pi z}{\ell_o}$$

But since n_o is a function of $\frac{a_o}{\ell_o}$ and $\frac{b_o}{b}$ and the material characteristics, the wave length will also depend on those values, besides showing special dependence on the fiber diameter.

A photomicrograph of residual stress in Epon 820 resin surrounding a smaller diameter E-glass single fiber is shown in Figure 4. Magnification is 100X. Fiber diameter was 0.0004 in. This free fiber was placed in a drop of resin and first cured at 250°F for 30 minutes. Examination at 75°F in polarized light following this cure showed no residual stress, and the fiber was straight. However, after an additional cure of 30 minutes at 350°F, the sample appeared as shown.

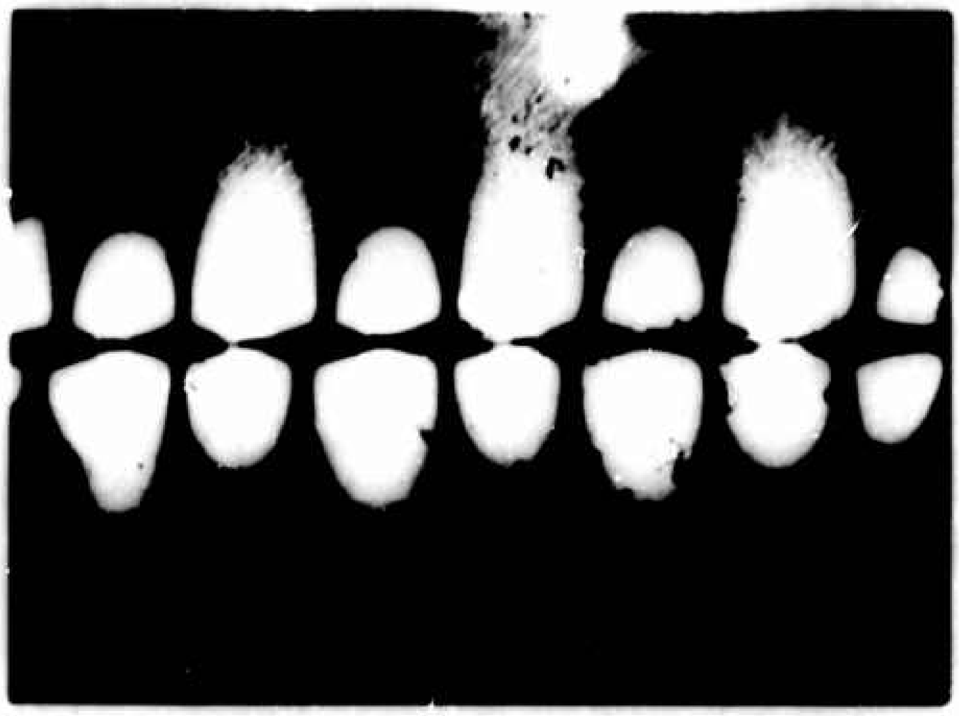

Figure 4. Photomicrograph of a Fine Fiber in Resin

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APPENDIX

DERIVATION OF FUNDAMENTAL EQUATIONS

The following is a brief discussion of the basic physical relations which will be required in later applications to a fiber reinforced matrix under various loads. In view of confusion in the literature about what is and what is not a tensor or tensor component, this introduction is considered necessary to define terminology.

First Order Theory Introduction

Consider the transformation from an orthogonal Cartesian coordinate system to a more general curvilinear system of coordinates. The transformation is a one-to-one transformation all of whose derivatives exist:

$$k^i = f^i(u^i), \quad i = 1, 2, 3 \tag{1}$$

Let $\vec{I}_i$ be a set of orthogonal unit vectors so that the scalar product gives

$$\vec{I}_i \cdot \vec{I}_j = G_{ij} \begin{array}{l} = 0, \ i \neq j \\ = 1, \ i = j \end{array} \tag{2}$$

e_{ijk} is defined by the scalar triple product (Reference 1, pages 16 and 20)

$$\vec{I}_i \times \vec{I}_j \cdot Ik = e_{ijk} \tag{3}$$

so that the vector product is given by

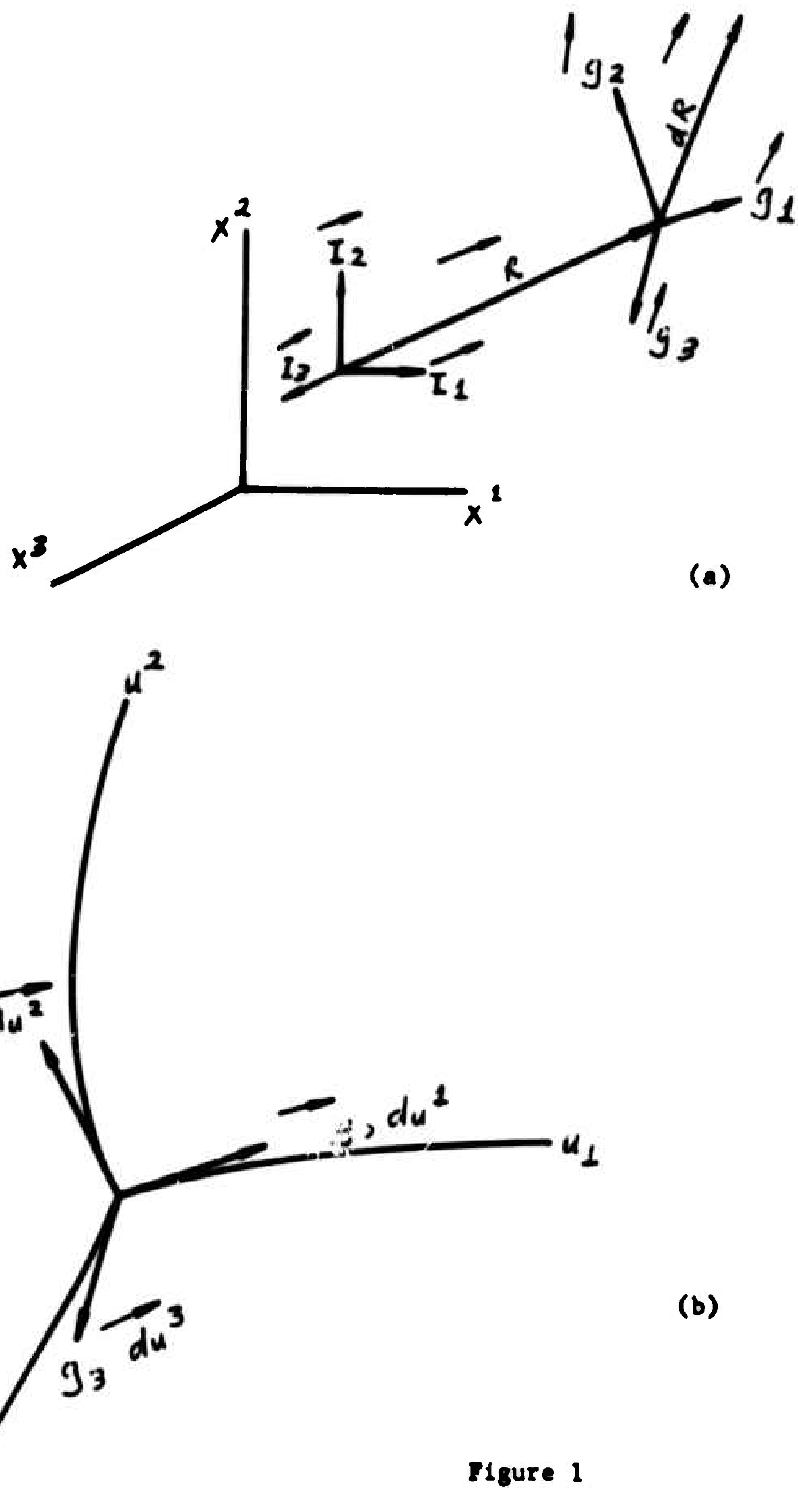
$$\vec{I}_i \times \vec{I}_j = e_{ijk} \vec{I}^k \tag{4}$$

e_{ijk} = 0, for any equal subscripts

= 1, for cyclic interchange of subscripts 1, 2, 3

= -1, for non-cyclic interchange of subscripts

The upper index $\vec{I}^k$ is used to conform to subsequent tensor notation.


x^2
$\bar{I}_2$
$\bar{R}$
$\bar{g}_2$
$d\bar{R}$
$\bar{g}_1$
$\bar{g}_3$
$\bar{I}_3$
$\bar{I}_1$
x^1
x^3
u^2
$\bar{g}_2\, du^2$
du^1
u_1
$\bar{g}_3\, du^3$
u^3

(a)

(b)

Figure 1

Consider the vector

$$\vec{R} = (x^i - x_v^i)\vec{I}_i \quad , \quad x^i = \text{Cartesian coordinates}$$

whose differential displacement is

$$d\vec{s} = d\vec{R} = dx^i \vec{I}_i$$

repeated indices summed which becomes under the transformation (1)

$$dx^i = \frac{\partial x^i}{\partial u^j} du^j \quad ; \quad du^j = \frac{\partial u^j}{\partial x^i} dx^i \tag{5}$$

$$d\vec{s} = d\vec{R} = du^j \frac{\partial x^i}{\partial u^j} \vec{I}_i = du^j \vec{g}_j \tag{6}$$

where

$$\vec{g}_j = \frac{\partial x^i}{\partial u^j} \vec{I}_i \quad ; \quad I_i = \frac{\partial u^j}{\partial x^i} \vec{g}_j \tag{7}$$

Equations (5) and (7) represent the contravariant and covariant transformations respectively. The vectors g_j are the metric vectors tangent tangent to the curvilinear coordinate elements du^j.

The differential arc length ds is given by the scalar product

$$ds^2 = d\vec{R} \cdot d\vec{R} = dx^i dx^j \vec{I}_i \cdot \vec{I}_j$$

$$= dx^i dx^j g_{ij}$$

$$= du^i du^j \vec{g}_i \cdot \vec{g}_j$$

$$= du^i du^j \vec{g}_i \cdot \vec{g}_j = du^i du_j \cdot g_{ij} \qquad (8)$$

where g_{ij}, the metric tensor is defined by

$$g_{ij} = \vec{g}_i \cdot \vec{g}_j$$

$$= G_{lm} \frac{\partial x^l}{du^i} \frac{\partial x^m}{\partial u^j} = \sum_l \frac{\partial x^l}{\partial u^i} \frac{dx^l}{\partial u^j} \qquad (9)$$

Although there is no numerical distinction between covariant tensor and contravariant tensor quantities in Cartesian coordinates, the maintenance of the upper and lower indices is a convenient accounting arrangement. For example, if the indices in equation (7) are raised, the contravariant metric vector system is

$$\vec{g}^j = \frac{\partial u^j}{\partial x^i} \vec{I}^i \;;\; \vec{I}^i = \frac{\partial x^i}{\partial u^j} \vec{g}^j \qquad (7a)$$

From these, one may define

$$g^{ij} = \vec{g}^i \cdot \vec{g}^j \qquad (10)$$

and show by equations (7), (7a), and (10) that

$$g^k \cdot g_j = \delta^k_j$$
$$g^{ki} g_{ij} = \delta^k_j \tag{10a}$$

By virtue of these equations and other properties of the metric tensor components g_{ij}, g^{ij}, these may be utilized for raising and lowering indices.

$$A^i = g^{ij} A_j \quad ; \quad A_j = g_{ij} A^j \tag{11}$$

Note from equation (9) that the absolute value of

$$|\vec{g}_i| = \sqrt{g_{ii}} \tag{12}$$

and hence

$$g_{oi} = \frac{\vec{g}_i}{\sqrt{g_{ii}}} \tag{13}$$

are unit vectors.

Differentiating equation (9),

$$\begin{aligned} \frac{\partial g_{ij}}{\partial u^k} &= \frac{\partial \vec{g}_i}{\partial u^k} \cdot \vec{g}_k + \vec{g}_j \cdot \frac{\partial \vec{g}_j}{\partial u^k} \\ &= G_{lm} \frac{\partial^2 x^l}{\partial u^k \partial u^i} \frac{\partial x^m}{\partial u^j} + G_{lm} \frac{\partial x^l}{\partial u^i} \frac{\partial^2 x^m}{\partial u^k \partial u^j} \end{aligned} \tag{14}$$

Interchanging i, j, k, adding and subtracting, yields

$$G_{\ell m} \frac{\partial^2 x^\ell}{\partial u^k \partial u^i} \frac{\partial x^m}{\partial u^j} = \frac{1}{2} \frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ji}}{\partial u^k} - \frac{\partial g_{ki}}{\partial u^j}$$

$$= [k i, j] \tag{15}$$

which is the Cristoffel symbol of the first kind. The Cristoffel symbol of the second kind is defined by

$$\left\{ {j \atop k i} \right\} = g^{j\ell} [k i, \ell] \tag{15a}$$

Also,

$$\frac{\partial \vec{g}_i}{\partial u^k} = \frac{\partial^2 x^\ell}{\partial u^k \partial u^i} \vec{I}_\ell = \frac{\partial^2 x^\ell}{\partial u^k \partial u^i} \vec{I}^m G_{\ell m}$$

$$= \frac{\partial^2 x^\ell}{\partial u^k \partial u^i} \frac{\partial x^m}{\partial u^j} \vec{g}^j G_{\ell m}$$

$$= [k i, j] \vec{g}^j$$

$$= \left\{ {j \atop k i} \right\} \vec{g}_j \tag{15b}$$

The left-hand side of this equation is defined, as rewritten, to be the covariant derivative:

$$\vec{g}_{i,j} = \frac{\partial \vec{g}_i}{\partial u^j} - \left\{ {k \atop i j} \right\} \vec{g}_k = 0 \tag{16}$$

In similar manner, the following can be shown:

$$\vec{g}^i_j = \frac{\partial \vec{g}^i}{\partial u^j} + \left\{ {i \atop j k} \right\} \vec{g}^k = 0 \tag{16a}$$

and from equation (14),

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial u^k} - \left\{ {l \atop ik} \right\} g_{lj} - \left\{ {l \atop jk} \right\} g_{il} = 0 \qquad (17)$$

in similar manner,

$$g^{ij}_{k} = \frac{\partial g^{ij}}{\partial u^k} + \left\{ {i \atop lk} \right\} g^{lj} + \left\{ {j \atop lk} \right\} g^{il} \qquad (17a)$$

so that the covariant derivations of the metric vectors and the metric tensor are zero.

The determinant of g_{ik} from equation (9), becomes

$$g = \| g_{ik} \| = \left\| \frac{\partial x^l}{\partial u^i} \frac{\partial x^l}{\partial u^k} \right\| = \left\| \frac{\partial x^l}{\partial u^i} \right\|^2 \qquad (18)$$

and by equation (7)

$$g = [\vec{g}_r \cdot \vec{g}_s \times \vec{g}_t]^2 \qquad (19)$$

Hence, by equation (3),

$$\vec{g}_r \cdot \vec{g}_s \times \vec{g}_t = \vec{g}_r \times \vec{g}_s \cdot \vec{g}_t = e_{rst} \sqrt{g} \qquad (20)$$

From these and the fact that $g^i \cdot g_j = \delta^i_j$,

$$\vec{g}_r \times \vec{g}_s = e_{rst} \sqrt{g}\ \vec{g}^t \qquad (21)$$

and likewise

$$\vec{g}^r \times \vec{g}^s = \frac{e^{rst}}{\sqrt{g}} \vec{g}_t \qquad (22)$$

where $\varepsilon^{rst} = \frac{e^{rst}}{\sqrt{g}}$ and $\varepsilon^{rst} = e_{rst} \sqrt{g}$

are tensors.

The area of the triangle determined by two of the three vectors

$$ds_i = g_i \, du^{(i)} \quad (i) \text{ not summed} \qquad (23)$$

is

$$\left.\begin{aligned} ds_k &= \frac{1}{2}\left[d\vec{s}_i \times d\vec{s}_j\right] du^i\, du_j \\ &= \frac{1}{2}\left[\vec{g}^k\right]\sqrt{g}\; du^i\, du^j \;, \quad i \pm j \pm k \\ &= \frac{1}{2}\sqrt{g g^{kk}}\; du^i\, du^j \end{aligned}\right\} \tag{24}$$

By equation (21),

$$g_o^k = \frac{\vec{g}^k}{\sqrt{g^{kk}}}$$

is a unit vector normal to the areal element dS_k ; hence,

$$\vec{n}\, dS = \sum_i \frac{g^i}{\sqrt{g^{ii}}}\, dS_i \tag{25}$$

where the sum of the vector areas of the faces of the tetrahedron, formed by the metric vectors $d\vec{S}_i$ is equal to the vector area of the fourth face $\vec{n}\, dS$, where $\vec{n}$ is the unit normal to the fourth face dS .

Setting

$$\vec{n} = n_i\, \vec{g}^i \tag{26}$$

one obtains

$$n_i \sqrt{g_{ii}}\; dS = dS_i \tag{26a}$$

First Order Strain

Consider a displacement vector

$$\vec{V} = V^i \vec{g}_i = V^i \sqrt{g_{ii}} \cdot \frac{\vec{g}_i}{g_{ii}} = \xi^i \vec{g}_{oi} \tag{27}$$

where the ξ^i are the physical components of the displacement vector $\vec{V}$, since $\vec{g}_{oi}$ are now unit vectors. ξ^i are not components of a tensor and are thus confined to a particular coordinate system. Taking the differential of equation (27)

$$d\vec{V} = \left(\frac{\partial V^i}{\partial u^j} \vec{g}_i + V^i \frac{\partial \vec{g}_i}{\partial u^j} \right) du^j \tag{28}$$

which by virtue of equation (15b) becomes

$$= \left(\frac{\partial V^i}{\partial u^j} + V^k \left\{ {i \atop kj} \right\} \right) \vec{g}_i \, du^j$$

and by equation (16)

$$= V^i_{;j} \, \vec{g}_i \, du^j$$

$$= V_{i,j} \, \vec{g}^i \, du^j$$

Because V_{ij} is not symmetric, the following may be written:

$$V_{i,j} = \frac{V_{i,j} + V_{j,i}}{2} + \frac{V_{i,j} - V_{j,i}}{2} \tag{29}$$

$$= D_{ij} + \omega_{ij}$$

so that the total displacement becomes

$$\vec{D} = \vec{V} + D_{ij} \, \vec{g}^i \, du^j + \omega_{ij} \, \vec{g}^i \, du^j \tag{30}$$

Here, the first and third terms represent a pure displaçement and a pure rotation respectively and the second term $D_{ij}\,\xi^{i}\,du^{j}$ represents the strain displacement where D_{ij} is the strain tensor. The rotation tensor ω_{ij} is not considered here, but may appear later where local couples give rise to nonzero values.

The Stress Tensor

Consider an elemental tetrahedron enclosed by the coordinate surfaces dS_i and an area dS, which is in equilibrium under forces acting across these faces; i.e.,

$$\vec{\sigma}\,dS = \sum_{i} \vec{\sigma}_i\, dS_i \tag{31}$$

where $\vec{\sigma}$ is the stress vector. $\vec{\sigma}$ is an invariant, but $-\vec{\sigma}_i$ are not, in the sense that they act across a particular set of coordinate face elements. From equations (26a) and (31),

$$\vec{\sigma} = \sum_{i} n_i\, \vec{\sigma}_i \sqrt{g_{ii}} \tag{32}$$

By the definition of n_i, equation (25), these are covariant components and hence, because of the invariance of $\vec{\sigma}$, the following may be written:

$$\vec{\sigma}_i \sqrt{g^{(ii)}} = \sigma^{ij}\,\vec{g}_j \tag{33}$$

which are now contravariant components of a tensor, and σ^{ij} is defined as the contravariant of the stress tensor.

From equations (32) and (33)

$$\vec{\sigma} = \sigma^{ij}\, n_i\, g_i \tag{34}$$

Also,

$$\vec{\sigma} = \sigma^{j} g_j \tag{35}$$

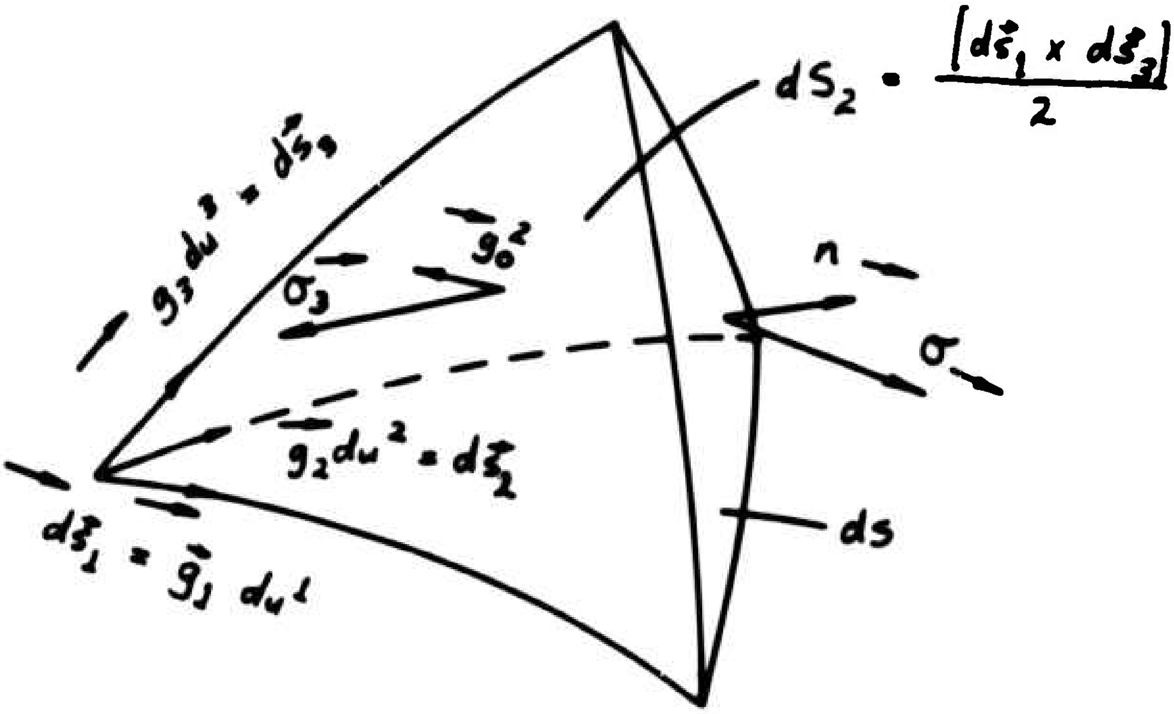

$dS_2 = \frac{|d\vec{s}_1 \times d\vec{s}_3|}{2}$
$\vec{g}_3 du^3 = d\vec{s}_3$
$\vec{g}_0^2$
$\vec{\sigma}_3$
$\vec{n}$
$\vec{\sigma}$
$\vec{g}_2 du^2 = d\vec{s}_2$
$d\vec{s}_1 = \vec{g}_1 du^1$
ds

Figure 2

so that

$$\vec{\sigma} = \sigma^{ij} n_i = g^j \cdot \vec{\sigma} \tag{36}$$

where σ^j are the contravariant vector components of the stress normal to dS_j .

The three stress vectors may be written (Reference 1, p. 78)

$$\vec{\sigma}_i = \sum_j \sqrt{\frac{g_{jj}}{g^{ii}}}\ \sigma^{ij}\ \frac{\vec{g}_j}{\sqrt{g_{jj}}} \tag{37}$$

$$= \sum_j \sigma^{ij}\ \vec{g}_{oj} \tag{38}$$

where σ^{ij} define the physical components of the stress tensor, but refer to a specific coordinate system and are not components of a tensor.

Equilibrium Relations

In Figure 3, the stress vectors acting across the face of the elements of the parallelopiped formed by the three vectors $\vec{g}_i\ du^{(i)}$ are, referring to Figure 3,

$$-\vec{\sigma}_i\ dS_{(i)} \quad ; \quad \vec{\sigma}_i\ dS_{(i)} + \frac{\partial\left(\vec{\sigma}_i\ dS_{(i)}\right)}{\partial u^i}\ du^{(i)} \tag{39}$$

or

$$-\vec{\sigma}_i \sqrt{g^{ii}}\ du^j\ du^k\ ;\ \vec{\sigma}_i\ dS_{(i)} + \frac{\partial\left(\vec{\sigma}_i\ g g^{ii}\right)}{\partial u^i}\ du^{(i)} du^j du^k$$

$$i \neq j \neq k$$

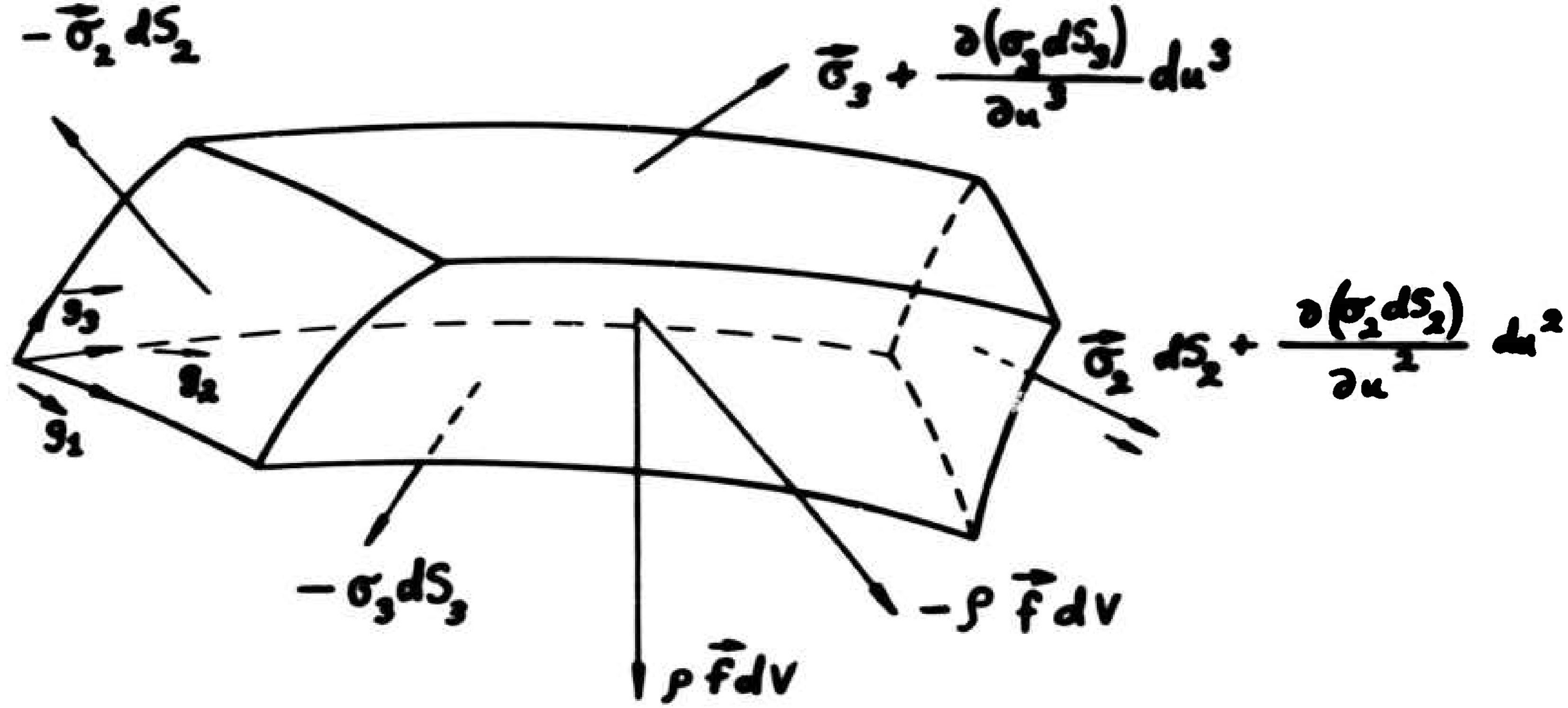

$-\vec{\sigma}_2\, dS_2$
$\vec{\sigma}_3 + \frac{\partial(\sigma_3 dS_3)}{\partial u^3} du^3$
$\vec{g}_3$
$\vec{g}_2$
$\vec{g}_1$
$\vec{\sigma}_2\, dS_2 + \frac{\partial(\sigma_2 dS_2)}{\partial u^2} du^2$
$-\sigma_3 dS_3$
$-\rho\, \vec{f}\, dV$
$\rho\, \vec{f} dV$

Figure 5

If no couples act on this element of volume, then from equation (39), keeping only first order terms,

$$\vec{g}_i \times \vec{\sigma}_i \sqrt{g^{ii}} = 0 \tag{40}$$

or from equations (37) and (22),

$$e_{ijk}\, \sigma^{ij}\, \vec{g}^k = 0 \tag{40a}$$

from which it follows that

$$\sigma^{ij} = \sigma^{ji} \tag{41}$$

and the stress tensor is symmetric.

The remaining equilibrium equation states that the sum of the forces acting on the element of volume are zero. Thus, if the mass acceleration of the volume is $-\rho \vec{f}\, dV$ and the body forces are $\rho \vec{F}\, dV$, where

$$\left.\begin{aligned} dV &= [g_1 \times g_2 \cdot g_3]\, du^1\, du^2\, du^3 \\ &= \sqrt{g}\, du^1\, du^2\, du^3 \end{aligned}\right\} \tag{42}$$

then the equilibrium of forces is expressed by

$$\sum_i \frac{\partial \sqrt{\sigma_i\, g g^{ii}}}{\partial u^i} + \rho F \sqrt{g} - \rho \vec{f} \sqrt{g} = 0 \tag{43}$$

or by equation (37)

$$\frac{\partial \sqrt{\sigma_i\, g g^{ii}}}{\partial u^i} + \rho F^j \sqrt{g}\, \vec{g}_j = \rho \vec{f} \sqrt{g} \tag{44}$$

which can be written

$$\frac{\partial(\sigma^{ij}\mathbf{g}_j)}{\partial u^i} + \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial u^i}\sigma^{ij}\mathbf{g}_j + \rho F^j\mathbf{g}_j = \rho f^j\mathbf{g}_j \quad (44a)$$

but

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial u^i} = \begin{Bmatrix} r \\ i\,r \end{Bmatrix}$$

and from equation (15b),

$$\frac{\partial\sigma^{ij}}{\partial u^i} + \begin{Bmatrix} j \\ i\,r \end{Bmatrix}\sigma^{ir} + \begin{Bmatrix} r \\ i\,r \end{Bmatrix}\sigma^{ij} + \rho F^j = \rho f^j \quad (45)$$

or

$$\sigma^{ij}_{,j} + \rho F^i = \rho f^i \quad (46)$$

Since σ^{ij} is a symmetric tensor, the contracted covariant derivative may be written in the form

$$\sigma^{\,i}_{j\,,i} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial u^i}\left(\sigma^{\,i}_{j}\sqrt{g}\right) - \frac{1}{2}\frac{\partial}{\partial u^i}(g_{\alpha\beta})D^{\alpha\beta} \quad (47)$$

which is much simpler to evaluate for a particular coordinate system.

Stress-Strain Relations

The conditions of equilibrium (41) and (45) furnish six relations for the determination of the nine stress tensor components. The three remaining are furnished through the physical equation functionally relating the stress and strain tensors. This functional relationship may be expressed

$$\sigma^{ij} = f^{ij}\left(D_{pq}\right) \quad (48)$$

where the functions are expendable in a Taylor's series. If this is expanded, retaining only first order terms, the result is a statement of Hooke's law.

$$\sigma^{ij} = \Lambda^{ij} + \Lambda^{ijpq} D_{pq} \tag{49}$$

or inversely

$$\left.\begin{aligned} D_{pq} - C_{pq} &= C_{pqij}\,\sigma^{ij} \\ D^p_q - C^p_q &= C^{pj}_{qi}\,\sigma^i_j \end{aligned}\right\} \tag{50}$$

where the constant terms C_{pq} , Λ^{ij} may represent initial stress-strain conditions due to polymeric shrinkage and differential thermal expansion.

The symmetry properties of a material are independent of coordinate choice, so that from the evaluation of the Hooke's law elastic tensor a Cartesian frame of reference may be chosen.

In isotropic media, such as the reinforcement or the matrix, the elastic tensor Λ^{ijpq} is invariant under rotation.

$$\Lambda^{ijpq} = A^p_i\, A^q_j\, A^r_k\, A^s_l\, \Lambda^{ijkl} \tag{51}$$

where A^i_j is a transformation of pure rotation and hence

$$\left.\begin{aligned} G^{ij} A^p_i\, A^q_j &= G^{pq} \\ G_{pq}\, A^p_i\, Q^q_j &= G_{ij} \end{aligned}\right\} \tag{52}$$

It follows that

$$A^{ijkl} = \lambda G^{ij} G^{kl} + \mu G^{ik} G^{jl} + \eta G^{il} G^{jk} \tag{53}$$

where λ, μ, η are scalar multipliers. In view of the symmetry of σ^{ij} and D_{ij}

$$\Lambda^{ijkl} = \Lambda^{jikl} = \Lambda^{ijlk} \qquad (54)$$

which leads to

$$\mu = \eta$$

and hence

$$\Lambda^{ijkl} = \lambda G^{ij} G^{kl} + \mu \left(G^{ik} G^{jl} + G^{il} G^{jk} \right) \qquad (55)$$

so that equation (49) becomes, placing $\Lambda^{ij} = 0$ where λ, μ are Lamé's constants

$$\sigma^{ij} = \left[\lambda g^{ij} g^{kl} + \mu \left(G^{ik} G^{jl} + G^{il} G^{jk} \right) \right] D_{kl} \qquad (56)$$

In generalised coordinates,

$$\sigma^{ij} = \left[\lambda g^{ij} g^{kl} + \mu \left(g^{ik} g^{jl} + g^{il} g^{jk} \right) \right] D_{kl} \qquad (57)$$

Placing

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \; ; \quad \mu = \frac{E}{1+\nu}$$

where E is Young's modulus of elasticity and ν is Poisson's ratio.

The following is obtained (Reference 1, p. 162):

$$\left.\begin{aligned}\sigma^{ij} &= \frac{E}{2(1+\nu)}\left[g^{ik}g^{j\ell} + g^{i\ell}g^{jk} + \frac{2\nu}{1-2\nu}\, g^{ij}g^{k\ell}\right]D_{k\ell}\\ \sigma^{ij} &= \frac{E}{1+\nu}\left(D^{ij} + \frac{\nu}{1-2\nu}\, g^{ij} D^{\ell}_{\ell}\right)\end{aligned}\right\} \quad (58)$$

or

$$\left.\begin{aligned}\sigma_{ij} &= \frac{E}{1+\nu}\left(D_{ij} + \frac{\nu}{1-2\nu}\, g_{ij}\, D^{\ell}_{\ell}\right)\\ \sigma^{i}_{j} &= \frac{E}{1+\nu}\left(D^{i}_{j} + \frac{\nu}{1-2\nu}\, \delta^{i}_{j}\, D^{\ell}_{\ell}\right)\end{aligned}\right\} \quad (59)$$

Inverting,

$$D^{ij} = \frac{1+\nu}{E}\,\sigma^{ij} - \frac{\nu}{E}\, g^{ij}\,\sigma^{\ell}_{\ell} \tag{60}$$

Temperature Expansion and Material Shrinkage

The constant terms in equation (49a) can be utilized to represent the internal effects of polymerization shrinkage and thermal expansion. If α represents the linear coefficient of thermal expansion and β the polymeric shrinkage, then

$$\begin{aligned} C^{i}_{j} &= g^{i}_{j}\,(\alpha T + \beta)\\ &= \delta^{i}_{j}\,(\alpha T + \beta)\end{aligned} \tag{61}$$

and

$$D^i_j - \delta^i_j (\alpha T + \beta) = \frac{1+\nu}{E} \sigma^i_j - \frac{\nu}{E} \delta^i_j \sigma^l_l \tag{62}$$

or inverting

$$\left.\begin{aligned} \sigma^i_j &= \frac{E}{1+\nu}\left\{D^i_j + \delta^i_j\left(\frac{\nu}{1-2\nu} D^l_l - \frac{1+\nu}{1-2\nu}\left[\alpha T + \beta\right]\right)\right\} \\ \sigma^{ij} &= \frac{E}{1+\nu}\left\{D^{ij} + g^{ij}\left(\frac{\nu}{1-2\nu} D^l_l - \frac{1+\nu}{1-2\nu}\left[\alpha T + \beta\right]\right)\right\} \end{aligned}\right\} \tag{63}$$

$$D^l_l = \sum_l D^l_l$$

__Differential Equations for the Displacement Vector__

Equation (46), for static applied forces, becomes

$$\sigma^{ij}_{,j} + \rho \bar{F} = 0 \tag{64}$$

From equation (63),

$$\sigma^{ij}_{,k} = \frac{E}{1+\nu}\left\{D^{ij}_{,k} + g^{ij}\left(\frac{\nu}{1-2\nu}\frac{\partial D^l_l}{\partial u^k} - \frac{1+\nu}{1-2\nu}\left[\alpha\frac{\partial T}{\partial u^k} + \frac{\partial \beta}{\partial u^k}\right]\right)\right\} \tag{65}$$

Contracting $\sigma^{ij}_{,k}$ to $\sigma^{ij}_{,i}$

$$\sigma^{ij}_{\ ,k} = \frac{E}{1+\nu}\left\{D^{ij}_{\ ,k} + g^{ij}\left(\frac{\nu}{1-2\nu}\frac{\partial D^{\ell}_{\ell}}{\partial u^{k}} - \frac{1+\nu}{1-2\nu}\left[\alpha\frac{\partial T}{\partial u^{k}} + \frac{\partial B}{\partial u^{k}}\right]\right)\right\} \quad (66)$$

Equation (64) now becomes

$$D^{ij}_{\ ,i} + g^{ij}\left\{\frac{\nu}{1-2\nu}\frac{\partial D^{\ell}_{\ell}}{\partial u^{i}} - \frac{1+\nu}{1-2\nu}\left[\alpha\frac{\partial B}{\partial u^{i}}\right]\right\} +$$

$$\rho \bar{F}^{j}\frac{1+\nu}{E} = 0 \quad (67)$$

or lowering indices

$$D^{i}_{j,i} + \frac{\nu}{1-2\nu}\left\{\frac{\partial D^{\ell}_{\ell}}{\partial u^{j}} - \frac{1+\nu}{1-2\nu}\left[\alpha\frac{\partial T}{\partial u^{j}} + \frac{\partial B}{\partial u^{j}}\right]\right\} +$$

$$\frac{1+\nu}{E}\rho \bar{F}_{j} = 0 \quad (68)$$

where $\bar{F}_1 = F_1 \cos\varphi + F_2 \sin\varphi$

$\bar{F}_2 = F_1\, r \sin\varphi + F_2\, r \cos\varphi$

$\bar{F}_3 = F_3$

$F_i = F^i$ are body forces in Cartesian coordinates

whereas $\bar{F}_i$ are in cylindrical coordinates.

Application to a Cylindrical Coordinate System

Let
$$\left.\begin{aligned} x^1 &= u^1 \cos u^2 , & u^1 &= r \\ x^2 &= u^1 \sin u^2 , & u^2 &= \varphi \\ x^3 &= u^3 , & u^3 &= z \end{aligned}\right\} \quad (69)$$

Then
$$\left.\begin{aligned} & g_{ij} = 0 , \quad i \neq j \\ & g_{11} = 1 , \quad g_{22} = (u^1)^2 , \quad g_{33} = 1 , \quad \sqrt{g} = r \\ & g_{11} = 1 , \quad g_{22} = r^2 , \quad g_{33} = 1 \end{aligned}\right\} \quad (70)$$

and
$$\left.\begin{aligned} & g^{ij} = 0 , \quad i \neq j \\ & g^{11} = 1 , \quad g^{22} = \frac{1}{(u^1)^2} , \quad g^{33} = 1 \\ & g^{11} = 1 , \quad g^{22} = \frac{1}{r^2} , \quad g^{33} = 1 \end{aligned}\right\} \quad (70a)$$

The only nonzero Cristoffel symbols of the second kind are

$$\left\{ {1 \atop 22} \right\} = -r$$

$$\left\{ {2 \atop 21} \right\} = \left\{ {2 \atop 12} \right\} = \frac{1}{r}$$

Further, by equation (49)

$$D_{ij} = \frac{1}{2}\left(g_{il}\, V^{l}_{,j} + g_{jl}\, V^{l}_{,i}\right)$$

$$= \frac{1}{2}\left(g_{(ii)} V^{i}_{,j} + g_{(jj)} V^{j}_{,i}\right)$$

where by equation (27)

$$V^{i} = \frac{\xi^{i}}{\sqrt{g_{ii}}}$$

Hence:

$$\left.\begin{aligned} D^{11} &= \frac{\partial \xi^{1}}{\partial r} \\ D^{22} &= \frac{1}{r}\left(\xi^{1} + \frac{\partial \xi^{2}}{\partial \varphi}\right) \\ D^{33} &= \frac{\partial \xi^{3}}{\partial z} \end{aligned}\right\} \qquad (71)$$

$$\left.\begin{aligned} D^{12} &= \frac{1}{2}\left[\frac{1}{r}\frac{\partial \xi^{1}}{\partial \varphi} + \frac{\partial \xi^{2}}{\partial r} - \frac{1}{r}\xi_{2}\right] \\ D^{23} &= \frac{1}{2}\left[\frac{\partial \xi^{2}}{\partial z} + \frac{1}{r}\frac{\partial \xi^{3}}{\partial \varphi}\right] \\ D^{31} &= \frac{1}{2}\left[\frac{\partial \xi^{3}}{\partial r} + \frac{\partial \xi^{1}}{\partial z}\right] \end{aligned}\right\} \qquad (72)$$

Utilizing equations (27) and (47) in equation (68)

$$\frac{1-\nu}{1-2\nu}\frac{\partial^2 \xi_1}{\partial r^2} + \frac{1}{2r^2}\frac{\partial^2 \xi_1}{\partial \varphi^2} + \frac{1}{2}\frac{\partial^2 \xi_1}{\partial z^2} + \frac{1}{2(1-2\nu)}\frac{1}{r}\frac{\partial^2 \xi_2}{\partial r \partial \varphi} +$$

$$\frac{1}{2(1-2\nu}\frac{\partial^2 \xi_3}{\partial r \partial z} - \frac{3-4\nu}{2(1-2\nu)}\frac{1}{r^2}\frac{\partial \xi_2}{\partial \varphi} +$$

$$\frac{1-\nu}{1-2\nu}\frac{1}{r}\left(\frac{\partial \xi_1}{\partial r} - \frac{1}{r}\xi_1\right) - \frac{1+\nu}{1-2\nu}\left(\alpha\frac{\partial T}{\partial r_1} + \frac{\partial \theta}{\partial r}\right) +$$

$$\frac{1+\nu}{E}\left(g_1 \cos\varphi - g_2 \sin\varphi\right) = 0 \quad (73)$$

$$\frac{1}{2}\frac{\partial^2 \xi_2}{\partial r^2} + \frac{1-\nu}{1-2\nu}\frac{1}{r^2}\frac{\partial^2 \xi_2}{\partial \varphi^2} + \frac{1}{2}\frac{\partial^2 \xi_2}{\partial z^2} +$$

$$\frac{1}{2(1-2\nu)}\frac{1}{r}\frac{\partial^2 \xi_1}{\partial r \partial \varphi} + \frac{1}{2(1-2\nu)}\frac{1}{r}\frac{\partial^2 \xi_3}{\partial \varphi \partial r} +$$

$$\frac{1}{2r}\left(\frac{1}{r}\frac{3-4\nu}{1-2\nu}\frac{\partial \xi_1}{\partial \varphi} + \frac{\partial \xi_2}{\partial r} - \frac{1}{r}\xi_2\right) - \frac{1+\nu}{1-2\nu}\frac{1}{r}\left(\alpha\frac{\partial T}{\partial \varphi} + \frac{\partial \theta}{\partial \varphi}\right) +$$

$$\left(g_1 \sin\varphi + g_2 \cos\varphi\right)\frac{1+\nu}{E} = 0 \quad (74)$$

$$\frac{1}{2}\frac{\partial^2 \xi_3}{\partial r^2} + \frac{1}{2r^2}\frac{\partial^2 \xi_3}{\partial \varphi^2} + \frac{1-\nu}{1-2\nu}\frac{\partial^2 \xi_3}{\partial z^2} + \frac{1}{2(1-2\nu)}\frac{\partial^2 \xi_1}{\partial r \partial z} +$$

$$\frac{1}{2(1-2\nu)}\frac{\partial^2 \xi_2}{\partial \varphi \partial z} + \frac{1}{2(1-2\nu)}\frac{1}{r}\frac{\partial \xi_1}{\partial z} + \frac{1}{2r}\frac{\partial \xi_3}{\partial r} -$$

$$\frac{1+\nu}{1-2\nu}\left(\alpha\frac{\partial T}{\partial z} + \frac{\partial \beta}{\partial z}\right) + g_3\frac{1+\nu}{E} = 0 \qquad (75)$$

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The mechanical interactions between reinforcement and matrix in a composite under load have been described by stress distribution equations in the fibers and in the matrix. Although special attention has been given to stresses resulting from polymerization and temperature shrinkage, the solutions obtained can be easily adapted for use with the boundary conditions of external loads. Several cases were considered: (1) a cylindrical filament of finite length embedded in a resin cylinder; (2) a cylindrical filament of infinite length embedded in a resin cylinder; and (3) a matrix supporting a central fiber surrounded by six symmetrically spaced fibers, each in turn surrounded by six others.

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14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
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